Scheduling Crackdowns on Illicit Drug Markets

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Abstract

This paper presents an analytical approach for scheduling crackdowns on street–corner drug markets. The crackdown scheduling problem is shown to be NP–Complete. A dynamic programming formulation is presented with an exponential time optimal algorithm. We then provide efficient optimal algorithms for several special cases and approximation algorithms for the general case. These results show that the optimal strategy is to give priority to markets that take longer to bring down and which require low levels of post-crackdown maintenance. The results are then extended to incorporate dealer displacement between drug markets.

1 Introduction

Drug abuse has persisted at a high level throughout the 1980s and into the 1990s, imposing significant costs on American society. In recent years U.S. drug control policy has given increasing attention to local drug enforcement. A currently favored local–enforcement strategy, "crackdowns," increases police activity in an area to increase the perceived and/or actual threat of apprehension [20]. The hope is that concentrating the resources in a geographic area will make them more effective. In effect, crackdowns are predicated on a belief in increasing returns to scale in enforcement.

Caulkins [5, 8] presents a quantitative model for predicting the rate of change of dealers with respect to time in a given market under focussed enforcement. Baveja et al. [4] analytically demonstrate in the context of Caulkins' model that the simple strategy of using the maximum available enforcement level until the market has collapsed minimizes the total effort which must be expended to collapse a market in most instances.

Most cities, however, have more than one street corner drug market. Past research addressed the question of the optimal rate at which to crack down on an individual market, but the strategic issue of specifying the order or sequence in which crackdowns should be initiated still needs to be considered. Often crackdowns are initiated at the insistence of neighborhood residents' demands to clean up the area. By this logic, city police chiefs typically initiate crackdown operations on the most severely affected neighborhoods. However, it is not clear that such a strategy is sound from a global perspective.

This paper provides an analytical framework for studying the *crackdown scheduling (CS)* problem. We formulate it as a scheduling problem and show that it belongs to the category of NP-hard problems. For the general crackdown scheduling problem, we present an exponential-time dynamic programming algorithm. Then, we make some realistic simplifying assumptions, and present efficient optimal solutions to the resulting special cases. We give approximation algorithms for the general case, and analyse their worst case performance. Lastly, we extend the crackdown scheduling problem to incorporate the dealer displacement effect.

The rest of the paper is organized as follows:

- Section 2 defines the problem.
- Section 3 establishes the problem to be computationally intractable by proving it to be NP-complete [12].
- Section 4 gives a dynamic programming formulation of the problem and an exponential-time optimal algorithm.
- Section 5 considers special cases of the problem. We give efficient optimal algorithms for these cases that suggest rules of thumb for policy makers.
- Section 6 presents approximation algorithms for the general problem.
- Section 7 extends the above formulation to incorporate dealer displacement between drug markets.
- Section 8 summarizes the important findings of this work.

2 Problem Definition

As discussed in the introduction, focusing efforts on particular geographically–defined markets is a commonly employed enforcement strategy. Typically crackdowns on individual markets are part of a longer–range plan which seeks to attack each market in turn. Hence, one can distinguish two types of resource allocation questions. First, at what rate should forces be deployed in a single market? Second, in what order should the markets in a city be attacked?

Previous modeling [4] of such focused enforcement operations concluded that when a fixed level of enforcement resources is available, the optimal strategy for eliminating an isolated market is (in almost all cases) to "Allocate the full amount of resources from the earliest instant." This simple strategy minimizes the integral of the amount of effort exerted over time, i.e., the total level of resources expended on the crackdown.

It might appear that this result renders the second problem moot. If the optimal strategy in individual markets is simply to apply all available resources, then it would seem that a collection of markets would be best addressed by attacking each in turn with all available resources. Furthermore, it would seem that the total amount of time, and hence the total amount of resources used, would be independent of the order in which the markets were attacked. For two reasons, however, the problem may not be separable in this fashion.

The first is that other theoretical work [5, 3] and practical experience [7] indicate that successful crackdown operations need to be followed by a maintenance phase to prevent the drug market from "springing back." The amount of effort needed varies with market size and other characteristics, but it is typically much less than the amount of effort needed to eliminate the market in the first place. Nevertheless, fewer resources will be available to attack the second market than the first because some of these resources will have to be left behind to protect the gains made in the first market.

The second problem is the possibility of displacement. It would be Pollyanish to imagine that all participants in a collapsed market immediately swear off all subsequent involvement in illicit drugs. Both dealers and users may seek other venues for their activities. Presumably these alternate venues are less desirable (e.g. because they are riskier); otherwise they would have been used previously. Hence, it is not implausible that there would be some decrease in total activity, but a total cessation is unlikely.

One can distinguish two kinds of displacement: displacement to other street markets and displacement to methods of operation which are less overt, less public. If the displacement is to so–called "quiet" dealing, this may both represent progress and circumvent the scheduling difficulty. Progress would be made because the level of negative externalities generated per unit sold are almost certainly higher for brazen, street–corner dealing than they are for surreptitious dealing that takes place behind closed doors [6]. It does not pose a problem for scheduling because the displacement does not directly affect the other street markets. The first half of this paper will address the case where the majority of displacement is to discreet dealing so displacement to other street markets can be ignored. Section 7 extends some of the results to the more general case in which there is displacement between street markets.

Here we seek to schedule crackdown operations, each of which uses the maximum available enforcement, in such a way that the total resources spent are the least. We define the crackdown scheduling (CS) problem as follows.

There exist k drug markets, $1, 2, \ldots, k$, all of which need to be brought down to a negligible level of drug activity. Let E represent the total level of resources allocated for this crackdown program at any instant of time. Each market is eliminated by a two step process: crackdown and maintenance phase. The crackdown phase utilizes all the *available* resources until the market is brought down to the desired level. This is followed by a maintenance phase which uses significantly smaller amount of resources to maintain the market at this desired level. Hence, each drug market i has the following two parameters:

1. A response function $f_i(X)$, giving the time necessary to bring down market i when $E - X$ units of effort are applied, i.e. when X units of effort are *unavailable*, but all

available effort is used. We assume that f_i is monotonically increasing in X since an increase in the amount of unavailable resources will necessarily result in an increase in the time needed to bring down the market.

2. A maintenance level w_i, which denotes the amount of resources needed to *maintain* market i at a negligible level of activity, once it is brought down. We assume no particular relationship between w_i and f_i.

The objective is to minimize the total effort expended in eliminating the k markets. Without loss of generality, effort expended maintaining gains after all of the markets have been suppressed is excluded from the objective function. Since E units of effort are employed at all times (counting both crackdown and maintenance allocations), the objective function reduces to minimizing the time until all markets have been eliminated. The time it takes depends on the order in which the focussed enforcement operations are carried out because the amount of resources available for cracking down on a market depends on the order in which the crackdown operations are initiated.

To illustrate this, consider a crackdown on the markets $1, \ldots, k$ in the order $1, 2, \ldots, k$. Then the cost of the operation on market 1 will be $E \times f_1(0)$. The cost of maintaining 1 will be w_1 multiplied by *the time to complete crackdown operations on markets* $2, \ldots, k$. For the second market, 2, the resources available for the crackdown operation will be $(E - w_1)$. The crackdown and maintenance costs for the second market will then be $(E - w_1) \times f_2(w_1) + w_2 \times$ (*the time to complete crackdown operations on the remaining markets,* $3, \ldots, k$.). Therefore, the total cost for the entire operation on the k drug markets will be the sum of costs of operations on individual markets, which simplifies to

$$E \times f_1(0) + E \times f_2(w_1) + \cdots + E \times f_k(\sum_{i=1}^{k-1} w_i) \quad (1)$$

Since our objective is to find an ordering that minimizes the total cost of the operation, the problem can be stated more formally as follows. Let $\Pi(i)$ denote the i^{th} element of the permutation Π of the indices $\{1, 2, \ldots, k\}$. Then the problem is to compute C^*, the value the cost function C,

$$C(\Pi) = \sum_{j=1}^{k} f_{\Pi(j)}(\sum_{l=1}^{j-1} w_{\Pi(l)}), \quad (2)$$

attains under a cost–minimizing schedule,

$$C^* \;=\; \min_{\Pi} \left[C(\Pi)\right]. \quad (3)$$

This statement of the problem, even though similar to some classes of well-known scheduling problems [2], is different in that the ordering impacts the cost function through the dependence of the function f_i as well as through the argument of the function. This property makes the problem difficult to analyze. In fact, as we will show in the following section, it belongs to the class of NP-hard problems, which are believed to be intractable.

Besides its useful application in this scenario, this problem is of interest as a natural generalization of existing scheduling problems which have been extensively studied [2].

3 Proof of NP-Completeness of Crackdown Scheduling

Here we show that the Crackdown Scheduling (CS) problem is NP-complete. We begin by redefining our problem as a decision problem.

Definition 1 *An instance of the Crackdown Scheduling decision problem has the following data as input:*

- *A set* $M = \{1, 2, \ldots, k\}$ *of drug markets and a constant* E *denoting the total police resources available,*
- *associated with every drug market* i*, a maintenance level* w_i *which denotes the amount of resources required to keep market* i *at the desired final state,*
- *a set of monotonically increasing* response functions, $\{f_i : 1 \leq i \leq k\}$ *such that* $f_i : \mathbf{R} \rightarrow \mathbf{R}$*, and* $f_i(X) = Y$*, if and only if it takes* Y *units of time to bring market* i *under control using* $(E - X)$ *resources, and*
- *a cost function, which gives the total cost of the crackdown operation if the markets are scheduled by the permutation* Π.

$$C(\Pi) \quad = \quad \sum_{j=1}^{k} f_{\Pi(j)}(\sum_{l=1}^{j-1} w_{\Pi(l)}).$$

The problem is as follows: Given an instance of the crackdown scheduling problem and a integer m*, is there a schedule* Π *of crackdowns on the given set of markets such that the cost function* $C(\Pi) \leq m$ *?*

Recall the definition of the minimum weighted tardiness problem, which is a well-known NP-complete problem [12].

Definition 2 *An instance of the minimum weighted tardiness problem is a set* $J = \{1, 2, \ldots, k\}$ *of jobs, such that job* i *has a fixed processing time* α_i*, a fixed deadline* β_i*, and a fixed weight* μ_i*. In any schedule, define the completion time of job* i *by* C_i*,* *and the* tardiness *of job i as $T_i = max\{0, C_i - \beta_i\}$. The tardiness of a schedule* Π *is given by* $C(\Pi) = \sum_{i=1}^{k} \mu_i T_i$.

The problem is as follows: Given an instance of a minimum weighted tardiness problem and an integer m, is there a schedule Π *such that the cost function* $C(\Pi) \leq m$*?*

We will show that every instance of the weighted tardiness problem reduces to an instance of CS, which will establish that CS is NP-complete, since the minimum weighted tardiness problem is known to be NP-complete [12].

Theorem 1 *The Crackdown Scheduling problem is NP-complete.*

Proof Consider an arbitrary instance I of the minimum weighted tardiness problem. From I, we will construct an instance I' of CS in polynomial time such that there exists a solution to I if and only if there exists a solution to I'.

For $1 \leq i \leq k$, let $\{\mu_i\}$ be the set of weights, $\{\alpha_i\}$ be the set of processing times and $\{\beta_i\}$ be the deadlines in I.

Define I', which is an instance of CS as follows. Let there be k markets, such that the response function for market i is given by the following expression.

$$f_i(X) = \begin{cases} (X + \alpha_i - \beta_i) \cdot \mu_i & \text{if } X + \alpha_i > \beta_i; \\ 0 & \text{otherwise.} \end{cases}$$

The maintenance level w_i of market i is given by $w_i = \alpha_i$. We need to show that there exists a solution to I if and only if there exists a solution to I'.

This follows from the observation that for the instances I and I' considered above, for any schedule Π, the value of $C(\Pi)$ for I is equal to the value of $C(\Pi)$ for I', from Equation 2, with the value of the response function f_i in the schedule Π taking the place of the tardiness

T_i of job i. Hence there is a schedule Π for which $C(\Pi) \leq m$ if and only if the value of $C(\Pi) \leq m$ for I', which completes the proof. $\square$

4 A Dynamic Programming Formulation

It is clear that the CS problem is inherently combinatorial. The search space of any naive algorithm trying to compute the optimal solution will be the number of permutations of the markets. We now give a dynamic programming formulation of the problem, which will not only help us understand the combinatorial structure of the problem, but also provide an exponential–time optimal algorithm. Since this problem belongs to the category of hard problems, polynomial time algorithms which compute the optimal solutions are unlikely to exist.

Here, we give an optimal algorithm which runs in $\mathcal{O}(k \cdot 2^k)$ time, which is much more efficient than a naive brute force search algorithm which can take up to $\mathcal{O}((k/e)^k)$ many steps to execute.

Given a set $1, 2, \ldots, k$ of drug markets with f_i and w_i as explained in the previous section, define $\mathrm{M} = \{1, 2, \ldots, k\}$, and let $S \subseteq \mathrm{M}$. Let $C(S)$ denote the minimum cost of conducting crackdown operations on the markets in S assuming that markets in the set $\mathrm{M} - S$ have already been scheduled for crackdowns. For any set S, the optimal value of the cost function $C(S)$ is obtained by taking each element $j \in S$, assuming that j is scheduled next, and then computing $C(S - \{j\})$ recursively. We take the minimum over all $j \in S$ to compute $C(S)$. Finally, we are interested in the value of $C(\mathrm{M})$.

$$\forall i \in \mathrm{M},\ C(\{i\}) \ = \ f_i(\sum_{j \in \mathrm{M} - \{i\}} w_j). \tag{4}$$

$$C(S) = \min_{j \in S} \left[f_j(\sum_{i \in \mathrm{M}-S} w_i) + C(S - \{j\}) \right]. \tag{5}$$

To compute $C(\mathrm{M})$ in the above formulation, we compute $C(S)$ for all sets $S \subseteq \mathrm{M}$ recursively using Equations 4 and 5. A more formal algorithm follows.

- For all $i \in \mathrm{M}$, compute $C(\{i\})$, using Equation 4.
- for $j = 2$ to k,

  Compute the value of $C(S)$ for all subsets S of M that are of size j, using Equation 5.
- Output $C(\mathrm{M})$.

Executing Equation 5 once takes $\mathcal{O}(k)$ steps, and there are $\mathcal{O}(2^k)$ many subsets of M. Hence the algorithm runs in $\mathcal{O}(k \cdot 2^k)$ steps.

This algorithm may be reasonable for small to medium size cities. For example, Hartford, Connecticut is believed to have about 20 drug markets. Thus, the algorithm will take on the order of $2^{20} \approx 10^6$ steps, which may be computationally feasible. However, Washington, D.C. is believed to have about 90 drug markets [13], for which the algorithm would require on the order of $2^{90} \approx 10^{27}$ steps, which is impractical. Therefore, we next consider some special cases for which more efficient algorithms exist.

5 Special Cases

In Section 3 we showed that the problem of scheduling crackdowns on various drug markets belonged to the category of NP-complete problems. Then in Section 4, we presented an optimal algorithm which had an exponential running time. Even though the formulation was mathematically interesting, the practical applicability of the algorithm for large cities was questionable.

At this point there are two directions one can follow to find results that are more practical for large cities. One would be to simulate scenarios and evaluate the performance of various heuristics in these scenarios. A recommendation based on such an analysis could provide an efficient way of getting *close to* optimal solutions. We take this approach in Section 6, where we suggest approximation algorithms for the CS problem and perform a worst case analysis of our algorithms.

The other way to proceed is to consider some special cases of the problem. The hope is that one can present analytical results for these special cases that may then be used as rules of thumb to assist policy makers.

In this section we therefore consider four simplifying assumptions in turn.

5.1 Post-Crackdown Maintenance by Other Agencies

Recall that a crackdown program typically includes two phases – the crackdown itself and a maintenance phase. Generally if the crackdown achieves its goal of largely eliminating drug dealing in the area, only a relatively modest amount of enforcement pressure is required to maintain those gains. This subsection considers the special case in which maintenance efforts draw none of the resources used for the crackdown, i.e., the special case in which E units of enforcement are available for all crackdowns irrespective of scheduling order.

This would be the case, for example, if the crackdowns were conducted by dedicated narcotics enforcement agents while the burden of maintenance was assigned to regular uniformed patrol units. It would also be the case if some of the maintenance were performed by people who are not involved in law enforcement at all. This may seem paradoxical at first, but the belief that the local citizenry perform as much if not more order maintenance as do professionals is gaining credence and underpins much of the philosophy of community policing [14]. Furthermore orderly communities, which hopefully include former drug markets at the end of a successful crackdown phase, are relatively resistant to incursions of drug dealing. Hence, it is not unreasonable to think of area residents, community groups, other city services such as sanitation, health and human services, recreation programs for youth, etc. as performing maintenance functions, which sometimes goes under the name of neighborhood reclamation. Indeed, such organizations were called on to perform some of the short term and much of the long term maintenance following crackdowns in Hartford, Connecticut [19].

As stated earlier, in these cases E units of resources are available for cracking down on all drug markets, irrespective of the sequence. So the time to bring market i under control is given by $f_i(0) = t_i$ and, as can be seen by substituting t_i into Equation 1, the total time required to bring the set of markets under control does not depend on the schedule. However, we are still interested in reducing the *total resources* used in the process. Earlier, minimizing total resources used reduced to minimizing total time because E units of resources were being used at every instant. In this case however, the total amount of resources, counting crackdown and maintenance resources, varies with the schedule of crackdowns.

Hence, we reevaluate the cost function in this changed scenario. For all i, $1 \leq i \leq k$, let $t_i = f_i(0)$. Then $T = \sum_{i \in \mathrm{M}} f_i(0)$ represents the total time required for the entire set of markets. To illustrate, let the markets be scheduled in the order $1, 2, \ldots, k$. Then, the total time for which market 1 has to be maintained is given by $(T - t_1)$, so the resources needed are given by $(T - t_1) \times w_1 + E \cdot t_1$. More generally, the cost function is:

$$C(\Pi) \quad = \quad (T - t_{\Pi(1)}) \cdot w_{\Pi(1)} + (T - t_{\Pi(1)} - t_{\Pi(2)}) \cdot w_{\Pi(2)} + \cdots + (T - \sum_{i=1}^{k} t_i) \cdot w_{\Pi(k)} + E \cdot T$$

$$
\begin{aligned}
= \quad & T \cdot (\sum_{i \in \mathrm{M}} w_i) - [t_{\Pi(1)} \cdot \sum_{i \in \mathrm{M}} w_i + t_{\Pi(2)} \cdot (w_{\Pi(2)} + \cdots + w_{\Pi(k)}) + \cdots \\
& + t_{\Pi(k)} \cdot w_{\Pi(k)}] + E \cdot T.
\end{aligned}
$$

Removing the constant term $E \cdot T$ and rearranging yields:

$$
C'(\Pi) = \left[\sum_{i=2}^{k} \left(t_{\Pi(i)} \cdot \sum_{j=1}^{i-1} w_{\Pi(j)} \right) \right]. \tag{6}
$$

We will show that it is optimal to schedule crackdowns in nonincreasing order of the ratio t_i/w_i. Before proving the result, let us justify the strategy intuitively. Consider the case when there are just two drug markets. If the time required to bring down both markets is the same, then the optimal strategy will be to crack down first on that market which requires a lesser effort to maintain. On the other hand, if the two markets require the same amount of resources per unit time to maintain, then the optimal strategy will be to schedule that market first which takes longer to bring down, since the market that is scheduled second has to be maintained for a shorter duration. This reasoning suggests cracking down first on markets with high values of t_i and low values of w_i. The following theorem verifies this intuition.

Theorem 2 *For all markets i, let $t_i = f_i(0)$. Suppose maintenance does not reduce the amount of resources available for cracking down on markets, so that E units of enforcement are available for each crackdown regardless of the order scheduled. Then, scheduling the markets in nonincreasing order of the ratio t_i/w_i yields an optimal solution.*

Proof Without loss of generality, we assume that the markets are labeled $1, 2, \ldots, k$ such that $\frac{t_1}{w_1} \geq \frac{t_2}{w_2} \geq \ldots \geq \frac{t_k}{w_k}$. The cost function is given by Equation 6.

$$
C'(\Pi) = t_{\Pi(2)} \cdot w_{\Pi(1)} + t_{\Pi(3)} \cdot (w_{\Pi(1)} + w_{\Pi(2)}) + \cdots + t_{\Pi(k)} \cdot \left(\sum_{i=1}^{k-1} w_{\Pi(i)} \right).
$$

This problem is equivalent to the familiar mean weighted flow time (MWFT) problem [18]. In an instance of MWFT, a set J of k jobs $J_1, J_2, \ldots, J_k$ is given, with job J_i having running time c_i and weight g_i. The problem is to find a schedule such that the weighted mean completion flow time is minimized. That is, for a schedule Π, the value of the cost function which is to be minimized is $C(\Pi) = \sum_{i \in \mathrm{J}} g_{\Pi(i)} \left(\sum_{j=1}^{i-1} c_{\Pi(j)} \right)$. [18] asserts that scheduling the jobs in nonincreasing order of the ratio $\frac{c_i}{g_i}$ yields an optimal solution. Here, market i plays the role of job J_i, t_i plays the role of g_i, and w_i plays the role of c_i. Hence, in this case, scheduling the jobs in the order $1, 2, \ldots, k$ yields an optimal solution. □

5.2 Equal Maintenance Requirements

This subsection considers the special case in which the resources deployed in the maintenance phase are the same for all markets. This special case may be relevant because of issues of equity, ignorance, and intimidation. Each of these justifications will be discussed in turn.

Policy makers allocating resources must deal with issues of equity as well as efficiency. By their very nature crackdowns are not equitable – some market(s) receive special treatment. The rationale for pursuing a program which is in some sense blatantly unfair is that the society as a whole benefits. Models of crackdowns [5, 3] and intuition suggest that the marginal benefit of deploying an additional unit of effort in a particular open–air market increases in the amount of effort already deployed.

It is these efficiency gains which justify unequal deployment of resources. Whether or not one finds these efficiency arguments very or only somewhat persuasive, they are almost certainly more palatable in the short run than in the long run. In the case of collapsing drug markets, the crackdown phase is typically short relative to the maintenance phase. Hence, it is relatively easy to imagine instances in which political considerations would necessitate deploying the same level of maintenance forces in all collapsed markets, irrespective of their original size.

Political arguments are not the only possible rationale for allocating equal maintenance forces across all markets. A second is ignorance. At this point we do not know enough about local markets to be able to look at one and decide how much maintenance pressure is enough. Hence, particularly given how much easier it is to maintain a market than it is to collapse one in the first place (or to collapse one that has "sprung back"), a cautious, but not unreasonable approach, is to leave behind maintenance forces that are presumed to be adequate even for the most stubborn markets.

Finally, the nature of intimidation may justify an equal maintenance pressure model. It may be that the level of maintenance pressure needed is more or less constant across markets, or at least there may be a common upper bound. Once dealing in a particular market has ceased, one police officer at all times may be enough to prevent a revival of the market for the simple reason that no one wants to be the first to make a deal. Whoever made the first public deal would presumably be arrested. Since dealing organizations are rarely organized [16, 10] enough to compensate an individual "worker" sufficiently to incur a very large risk of arrest and incarceration, no dealer (or buyer) would be willing to be the first, so an empty market would be stable.

Hence for a variety of reasons, it is interesting to study the case in which the maintenance level is the same in all markets. In this scenario the crackdown scheduling problem reduces to a weighted bipartite matching problem and, hence, can be solved efficiently. A crackdown scheduling problem with equal maintenance levels requires that we find a permutation Π that minimizes $\sum_{i=1}^{k} f_{\Pi(i)}((i-1)w)$. It is convenient to formulate this case as a problem in graph theory. First, we discuss the necessary preliminaries.

A graph $G = (V, E)$ is bipartite if its vertex set V can be partitioned into two sets V_1 and V_2 such that all edges in G connect some element in V_1 to some element in V_2. A bipartite graph is denoted by $G = (V_1, V_2, E)$. Given an undirected graph $G = (V, E)$, a *matching* is a subset of edges $M \subseteq E$ such that for all vertices $v \in V$, at most one edge of M is incident on v. A maximal matching is a matching such that adding any edge to the matching set destroys the matching property of the set. The graph G is a weighted graph if each edge has a positive number associated with it. The minimum weighted matching problem is defined as follows: Given a weighted graph $G = (V, E)$, find a maximal matching of G which has minimum weight.

Given an instance $1, 2, \ldots, k$ of drug markets, construct a complete weighted bipartite graph $G^* = (V_1, V_2, E)$ as follows. V_1 corresponds to the set of markets, while V_2 is the order $1, 2, \ldots, k$. For all nodes $v_i \in V_1$ and $v_j \in V_2$, the weight of (v_i, v_j) is equal to the value of $f_i((j-1)w)$.

In other words, an edge (v_i, v_j) gives the cost added when i is scheduled at stage j. So, the problem now becomes, how do we choose k edges from E such that the sum of the edge weights is minimum, subject to the constraint that no two edges are incident to the same vertex? This is equivalent to solving the minimum weighted bipartite matching problem for G^*, and efficient algorithms are known for this problem [18].

5.3 Equity of Response Functions

In Section 5.1 we showed that if the burden of maintaining a market after a crackdown does not reduce the amount of resources available for conducting crackdowns, then the optimal strategy is to schedule crackdowns on markets in nonincreasing order of $\frac{f_i(0)}{w_i}$. We now consider a situation in which all the drug markets respond similarly to enforcement pressure. That is, the response function f_i for each market is the same.

Under such a scenario, the maintenance force for each market is the only variable parameter which affects the cost function directly. Intuitively one should schedule that market which requires the least maintenance first since it will leave the most resources available for markets to be scheduled later. Likewise the market requiring the next to least amount of maintenance effort should be scheduled second, and so on. This is consistent with the result from Section 5.1 where we showed that scheduling of markets with lower maintenance force first is desirable. Below we prove the correctness of this intuition.

Theorem 3 *If $f_1 \equiv f_2 \equiv \ldots \equiv f_k \equiv f$ for a given set of k markets, then scheduling crackdowns in order of nondecreasing w_i minimizes the total resources expended.*

Proof If all the response functions f are identical, then the cost function has the following form:

$$C(\Pi) = f(w_{\Pi(1)}) + f(w_{\Pi(1)} + w_{\Pi(2)}) + \cdots + f(\sum_{i \in \mathrm{M}-k\}} w_{\Pi(i)}). \tag{7}$$

Without loss of generality, we label the markets such that $w_1 \leq w_2 \leq w_3 \leq \cdots \leq w_k$. Suppose there exists an optimal schedule Π such that for some i, $w_{\Pi(i)} > w_{\Pi(i+1)}$. Consider the schedule $\tilde{\Pi}$ which is identical to Π except that it interchanges the order of markets $\Pi(i)$ and $\Pi(i+1)$. Then we have, from Equation 7,

$$C(\Pi) - C(\tilde{\Pi}) = f(q + w_{\Pi(i)}) - f(q + w_{\Pi(i+1)}) \tag{8}$$

where $q = \sum_{j=1}^{i-1} w_{\Pi(j)}$. Since $w_{\Pi(i)} > w_{\Pi(i+1)}$ and f is an increasing function, we have $C(\Pi) - C(\tilde{\Pi}) > 0$, which is a contradiction. This completes the proof. $\square$

5.4 Linear Response Functions

In this subsection we consider the special case in which f_i is linear for all i. The exact shape of these response functions is not known of course, but the general shape suggested by Caulkins [5] is that of a positive, increasing, convex function. The function should go off to infinity at some finite point because when enough resources are *unavailable*, the market will never collapse, no matter how long the crackdown goes on. If this singularity is relatively close to the origin, that is, even before allocating any resources for maintenance, the total resources available are barely sufficient to collapse the market, then obviously all the f_is are not linear. If, on the other hand, E is large, say several times the amount of enforcement necessary to collapse the most stubborn market even after maintenance forces have been dispatched to the other markets, then $f_i(X)$ for $X \leq \sum_{j=1}^{k} w_j$ will be relatively flat for all i, and hence a linear approximation $f_i(X) = m_i \cdot X + c_i$ may not be too bad.

We assume that if X is the amount of resources that are unavailable when market i is being cracked down upon, then the time taken for market i to be controlled is given by $f_i(X) = m_i \cdot X + c_i$, where m_i and c_i are positive real numbers. Note that the contribution of the c_is to the cost function does not depend on the position of i in the schedule; hence, they can be ignored.

We will now show that a strategy that schedules the markets in nonincreasing order of the ratio $\frac{m_i}{w_i}$ produces an optimal solution.

Theorem 4 *Let $\{1, 2, \ldots, k\}$ be a given set of markets with maintenance levels $\{w_i\}$ and the response functions $f_i(X) = m_i \cdot X + c_i$. Then a schedule Π with the property that*

$$\frac{m_{\Pi(1)}}{w_{\Pi(1)}} \geq \frac{m_{\Pi(2)}}{w_{\Pi(2)}} \geq \cdots \geq \frac{m_{\Pi(n)}}{w_{\Pi(n)}}$$

is an optimal schedule.

Proof Let Π be a schedule with the property given above. Suppose there exists an optimal schedule Π_1 such that for some i, $\frac{m_{\Pi_1(i)}}{w_{\Pi_1(i)}} < \frac{m_{\Pi_1(i+1)}}{w_{\Pi_1(i+1)}}$. Consider the schedule $\tilde{\Pi}_1$ which is identical to Π_1 except that it interchanges the order of markets $\Pi_1(i)$ and $\Pi_1(i+1)$. Then, on substituting values of f_i into Equation 2, we have:

$$C(\Pi_1) - C(\tilde{\Pi}_1) = m_{\Pi_1(i+1)} \cdot w_{\Pi_1(i)} - m_{\Pi_1(i)} \cdot w_{\Pi_1(i+1)}. \tag{9}$$

Since $\frac{m_{\Pi_1(i+1)}}{w_{\Pi_1(i+1)}} > \frac{m_{\Pi_1(i)}}{w_{\Pi_1(i)}}$, $C(\Pi_1) - C(\tilde{\Pi}_1) > 0$, which is a contradiction. This completes the proof. □

Intuitively, what this strategy says is that one should start with those markets which are the hardest to control but easiest to maintain once they are suppressed.

6 Approximation Algorithms

In this section we give approximation algorithms for the general case of the crackdown scheduling problem. First we consider crackdown scheduling on drug markets with polynomial response functions and suggest three strategies which yield "close-to-optimal" solutions. We also derive bounds on the performance of these algorithms.

Then we present two heuristics: the first one is based on the algorithm by Fisher and Krieger [11]. This heuristic is applicable to any instance of crackdown scheduling with convex response functions (not necessarily polynomials); the second heuristic is an adaptation of the algorithm by Alidaee [1], and can be applied on markets which have differentiable response functions.

The framework of the crackdown scheduling problem is complementary to that of the single machine scheduling problem which maximizes profits [11]. While approximation algorithms for the maximization problem yield solutions within 2/3 of the optimal [11], no bounds are known for the corresponding minimization problem [9]. Therefore, the performance bounds for our algorithms are also of value in single machine scheduling problems (e.g., the problem studied in [1]). We also suggest an alternate maximization formulation of the crackdown scheduling problem for which an adaptation of the Fisher–Krieger approximation algorithm ([11]) yields solutions which are within 2/3 of the optimal.

6.1 Polynomial Response Functions

One of the arguments for crackdowns is that the marginal benefit of deploying additional resources increases in the amount of effort already deployed [8]. Baveja et al. [3] show that one aspect of this increasing returns to scale is that the response functions are convex. Simple polynomials of the form $f(X) = m \cdot X^{\beta} + c$ for some constant $\beta \geq 1$ can approximate many convex functions.

We assume that if X is the amount of resources that are *not* available when market i is being cracked down upon, then the time taken for market i to be controlled is given by $f_i(X) = m_i \cdot X^{\beta} + c_i$, where m_i and c_i are positive reals, and c_i indicates the amount of time needed to control market i when all of the resources, E, are being used. Note that the contribution of the c_is to the cost function is independent of the position of i in the schedule. So, all the c_is are just additive constants in the expression for the cost function, and they do not appear in the strategy.

In this section we present three approximation algorithms that yield objective function values within $(k-1)^{\beta-1}$, $\frac{m_{max}}{m_{min}}$, and $\frac{w_{max}^{\beta}}{w_{min}^{\beta}}$, respectively, of the optimal solution value, C^*. All these algorithms run in time $\mathcal{O}(k \log k)$. By running all three algorithms and taking the schedule which yields the minimum cost, it is possible to obtain a performance bound of

$\min\{(k-1)^{\beta-1}, \frac{m_{max}}{m_{min}}, \frac{w^{\beta}_{max}}{w^{\beta}_{min}}\}$.

The appendix shows an efficient optimal algorithm for solving the Crackdown Scheduling problem when the response functions of the markets are polynomials, and a certain "agreeability" assumption on the drug markets is satisfied.

6.1.1 A $(k-1)^{\beta-1}$-bound algorithm

In this section we present an $\mathcal{O}(k \log k)$ time algorithm which yields a solution which is at worst $(k-1)^{\beta-1}$ away from the optimal objective function value.

Theorem 5 *Suppose I is an arbitrary instance of CS with k markets, such that market i has maintenance level w_i and response function $f_i(X) = m_i \cdot X^{\beta}$. Assume that:*

$$\frac{m_1}{w_1^{\beta}} \geq \frac{m_2}{w_2^{\beta}} \geq \dots \geq \frac{m_k}{w_k^{\beta}}.$$

Then $\Pi_1 = 1, 2, 3, \dots, k$ is a schedule such that

$$\frac{C(\Pi_1)}{C^*} \leq (k-1)^{\beta-1}.$$

Proof Let I be as stated in the hypothesis. Using I, define another instance I^{β} of CS as follows: I^{β} has k drug markets $1, 2, \dots, k$ such that market i has a maintenance level w_i^{β} and response function $g_i(X) = m_i \cdot X$. Since I^{β} is an instance of CS with linear response functions, it can be solved optimally in polynomial time. By the result in Section 5.4, it follows that $\Pi_1 = 1, 2, \dots, k$ is an optimal solution of I^{β}. We will now show that the schedule Π_1 solves I to within $(k-1)^{\beta-1}$ of C^*.

By definition of C^*:

$$C^* = \min_{\Pi} \left[\sum_{i=2}^{k} m_{\Pi(i)} \cdot \left(\sum_{j=1}^{i-1} w_{\Pi(j)} \right)^{\beta} \right].$$

Expanding the summation terms in the above expression, we have:

$$\begin{aligned} C^* &= \min_{\Pi} \left[\sum_{i=2}^{k} m_{\Pi(i)} \cdot \left(\sum_{j=1}^{i-1} w_{\Pi(j)^\beta} \right) + \alpha_{\Pi}(k) \right]. \\ &\geq \min_{\Pi} \left[\sum_{i=2}^{k} m_{\Pi(i)} \left(\sum_{j=1}^{i-1} w_{\Pi(j)}^{\beta} \right) \right] + \min_{\Pi} [\alpha_{\Pi}(k)] \\ &\geq \min_{\Pi} \left[\sum_{i=2}^{k} m_{\Pi(i)} \left(\sum_{j=1}^{i-1} w_{\Pi(j)}^{\beta} \right) \right] \qquad (10) \end{aligned}$$

where $\alpha_{\Pi}(k)$ is a positive number which is equal to the value of the remaining terms in the expansion. Now consider the expression which gives the optimal cost for I^β. The optimal cost $\overline{C}$ for I^β is:

$$\begin{aligned} \overline{C} &= \min_{\Pi} \left[\sum_{i=2}^{k} m_{\Pi(i)} \cdot \left(\sum_{j=1}^{i-1} w_{\Pi(j)}^{\beta} \right) \right] \\ &= m_2 \cdot w_1^\beta + m_3 \cdot (w_1^\beta + w_2^\beta) + \cdots + m_k \cdot (\sum_{j=1}^{k-1} w_j^\beta) \qquad (11) \end{aligned}$$

By Equations 10 and 11, it follows that

$$C^* \geq m_2 \cdot w_1^\beta + m_3 \cdot (w_1^\beta + w_2^\beta) + \cdots + m_k \cdot (\sum_{i=1}^{k-1} w_i^\beta). \qquad (12)$$

The value of $C(\Pi_1)$ can be written as:

$$C(\Pi_1) = m_2 \cdot w_1^\beta + m_3 \cdot (w_1 + w_2)^\beta + \cdots + m_k \cdot (\sum_{i=1}^{k-1} w_i)^\beta. \qquad (13)$$

From Equations 12 and 13 we have:

$$\frac{C(\Pi_1)}{C^*} \leq \frac{m_2 \cdot w_1^\beta + m_3 \cdot (w_1 + w_2)^\beta + \cdots + m_k \cdot (\sum_{i=1}^{k-1} w_i)^\beta}{m_2 \cdot w_1^\beta + m_3 \cdot (w_1^\beta + w_2^\beta) + \cdots + m_k \cdot (\sum_{i=1}^{k-1} w_i^\beta)} \qquad (14)$$

The theorem follows from the following lemma.

Lemma 1

$$\frac{C(\Pi_1)}{C^*} \leq \frac{m_2 \cdot w_1^\beta + \cdots + m_k \cdot (\sum_{i=1}^{k-1} w_i)^\beta}{m_2 \cdot w_1^\beta + m_3 \cdot (w_1^\beta + w_2^\beta) + \cdots + m_k \cdot (\sum_{i=1}^{k-1} w_i^\beta)} \leq (k-1)^{\beta-1}.$$

Proof of Lemma We establish an upper bound on Equation 14 by keeping the numerator fixed and by minimizing the denominator. To do this, we fix each term of the form $(w_1 + w_2 + \cdots + w_i)^\beta$ in the numerator, and minimize the corresponding term in the denominator, which is of the form $w_1^\beta + w_2^\beta + \cdots + w_i^\beta$. For all terms in the above form in the numerator, we look at the corresponding minimization problem for the denominator. In other words, we solve the following problem: Let A_i be a constant.

$$\begin{aligned} &Given \;\; w_1 + w_2 + w_3 + \cdots + w_i \;=\; A_i \\ &Minimize \;\; w_1^\beta + w_2^\beta + \cdots + w_i^\beta \end{aligned}$$

It can be shown that the solution to the above subproblem is the following:

$$w_1 = w_2 = \ldots = w_i = w = \frac{\sum_{j=1}^{i} w_j}{i}.$$

Substituting the above solution into Equation 14, we get,

$$\begin{aligned} \frac{C(\Pi_1)}{C^*} &\leq \frac{m_2 \cdot w^\beta + m_3 \cdot (w+w)^\beta + \cdots + m_k \cdot ((k-1) \cdot w)^\beta}{m_2 \cdot w^\beta + m_3 \cdot 2w^\beta + \cdots m_k \cdot (k-1)w^\beta} \\ &\leq \frac{m_2 + 2^\beta m_3 + \cdots + (k-1)^\beta m_k}{m_2 + 2m_3 + \cdots + (k-1)m_k} \end{aligned} \tag{15}$$

Now, with variables $m_1, m_2, \ldots, m_k$, we try to minimize the denominator, by solving the following minimization problem: Let B be a constant.

$$\begin{aligned} &Given \;\; m_2 + 2^\beta m_3 + \cdots + m_k(k-1)^\beta \;=\; B \\ &Minimize \;\; m_2 + 2m_3 \cdots + (k-1)m_k. \end{aligned}$$

The solution to this minimization problem is: $m_2 = m_3 = \ldots m_{k-1} = 0$, and $m_k = B/[(k-1)^\beta]$. Substituting this solution in Equation 15, we have:

$$\begin{aligned}\frac{C(\Pi_1)}{C^*} &\leq \frac{(k-1)^\beta m_k}{(k-1)m_k} \\ &= (k-1)^{\beta-1}.\end{aligned}$$

This completes the proof of the lemma. □

The theorem now follows immediately. □

6.1.2 An $\frac{m_{max}}{m_{min}}$-approximation algorithm

Here we present another $\mathcal{O}(k \log k)$-time algorithm for the crackdown scheduling problem with polynomial response functions. For any input instance I, let $m_{max} = \max_i\{m_i\}$ and $m_{min} = \min_i\{m_i\}$. Then, this algorithm yields an objective function value which is at most $\frac{m_{max}}{m_{min}}$ of the optimal.

Theorem 6 *Suppose I is an arbitrary instance of* CS *with k markets, such that market i has maintenance level w_i and response function $f_i(X) = m_i \cdot X^\beta$. Assume that:*

$$w_1^\beta \leq w_2^\beta \leq \ldots \leq w_k^\beta.$$

Then $\Pi_1 = 1, 2, 3, \ldots, k$ is a schedule such that

$$\frac{C(\Pi_1)}{C^*} \leq \frac{m_{max}}{m_{min}}.$$

Proof Consider an instance I' of CS with k markets $1, 2, \ldots, k$ such that the maintenance level and the response function for market i is w_i and $f(X) = m \cdot X^\beta$ respectively, where $m = m_{max}$. I' is an instance of CS with identical response functions, and by the result in

Section 5.3, it follows that the schedule $1, 2, 3, \ldots, k$ is an optimal solution of I'. So, if $D(\Pi)$ is the value of the cost function for I' with the schedule Π then:

$$\min_{\Pi} D(\Pi) \;=\; m \cdot (w_1^\beta + (w_1 + w_2)^\beta + \cdots + (\sum_{j=1}^{k-1} w_j)^\beta). \tag{16}$$

The value of C^* is given by the following expression.

$$\begin{aligned} C^* &= \min_{\Pi} \left[\sum_{i=2}^{k} m_{\Pi(i)} \cdot \left(\sum_{j=1}^{i-1} w_{\Pi(j)} \right)^\beta \right] \\ &\geq m_{min} \cdot \left[\min_{\Pi} \left(w_{\Pi(1)}^\beta + (w_{\Pi(1)} + w_{\Pi(2)})^\beta + \cdots (\sum_{j=1}^{k-1} w_{\Pi(j)})^\beta \right) \right] \\ &= m_{min} \left(w_1^\beta + (w_1 + w_2)^\beta + \cdots + (\sum_{j=1}^{k-1} w_j)^\beta \right) \end{aligned} \tag{17}$$

The value of the cost function $C(\Pi_1)$ can be computed as follows:

$$\begin{aligned} C(\Pi_1) &= m_2 \cdot w_1^\beta + m_3 \cdot (w_1 + w_2)^\beta + \cdots + m_k \cdot (\sum_{i=1}^{k-1} w_i)^\beta \\ &\leq m_{max} \left(w_1^\beta + (w_1 + w_2)^\beta + \cdots + (\sum_{j=1}^{k-1} w_j)^\beta \right). \end{aligned} \tag{18}$$

From Equations 17 and 18, it follows that:

$$\frac{C(\Pi_1)}{C^*} \;\leq\; \frac{m_{max}}{m_{min}}.$$

This completes the proof. □

6.1.3 A $\frac{w_{max}^\beta}{w_{min}^\beta}$-approximation algorithm

We now give an approximation algorithm which yields a solution within $\frac{w_{max}^\beta}{w_{min}^\beta}$ of the optimal where $w_{max} = \max_i \{w_i\}$ and $w_{min} = \min_i \{w_i\}$.

Theorem 7 *Suppose I is an arbitrary instance of CS with k markets, such that market i has maintenance level w_i and response function $f_i(X) = m_i \cdot X^{\beta}$. Assume that:*

$$m_1 \geq m_2 \geq \ldots \geq m_k.$$

Then $\Pi_1 = 1, 2, 3, \ldots, k$ is a schedule such that

$$\frac{C(\Pi_1)}{C^*} \leq \frac{w^{\beta}_{max}}{w^{\beta}_{min}}.$$

To prove Theorem 7 it is convenient to first prove the following lemma.

Lemma 2 *Consider an instance I' is CS such that I' has k markets, and the maintenance level for market i is $w = w_{max}$ and the response function is $f_i(X) = m_i \cdot X^{\beta}$. Then if the markets are labeled such that:*

$$m_1 \geq m_2 \geq \ldots \geq m_k$$

then the schedule $\Pi = 1, 2, \ldots, k$ is an optimal schedule.

Proof of Lemma The objective function of I' can be written as:

$$\begin{aligned} D(\Pi) &= m_{\Pi(2)} \cdot w^{\beta} + m_{\Pi(3)} \cdot (w+w)^{\beta} + \cdots + m_{\Pi(k)} \cdot ((k-1) \cdot w)^{\beta} \\ &= w^{\beta} \cdot \left[m_{\Pi(2)} \cdot 1^{\beta} + m_{\Pi(3)} \cdot 2^{\beta} + \cdots + m_{\Pi(k)} \cdot (k-1)^{\beta} \right]. \end{aligned} \tag{19}$$

We use an adjacent pairwise interchange argument similar to the previous proofs to prove this lemma. Suppose there exists an optimal schedule Π^* such that for some i, $m_{\Pi^*(i)} < m_{\Pi^*(i+1)}$. Consider the schedule $\overline{\Pi^*}$ which is identical to Π^* except that it interchanges the order of markets $\Pi^*(i)$ and $\Pi^*(i+1)$. Then, on substituting values of $\overline{\Pi^*}$ into Equation 19, we have:

$$\begin{aligned} D(\Pi^*) - D(\overline{\Pi^*}) &= m_{\Pi^*(i+1)} \cdot i^{\beta} - m_{\Pi^*(i)} \cdot (i-1)^{\beta} + m_{\Pi^*(i)} \cdot i^{\beta} - m_{\Pi^*(i+1)} \cdot (i-1)^{\beta} \\ &= i^{\beta} \cdot (m_{\Pi(i+1)} - m_{\Pi(i)}) - (i-1)^{\beta}(m_{\Pi(i+1)} - m_{\Pi(i)}) \end{aligned}$$

Since $m_{\Pi^*(i+1)} > m_{\Pi^*(i)}$, and $i^\beta > (i-1)^\beta$, it follows that $D(\Pi^*) - D(\overline{\Pi^*}) > 0$, which is a contradiction. This proves the lemma. □

Proof of Theorem For the given instance I, the value of C^* is given by the following expression.

$$\begin{aligned} C^* &= \min_{\Pi} \left[\sum_{i=2}^{k} m_{\Pi(i)} \cdot \left(\sum_{j=1}^{i-1} w_{\Pi(j)} \right)^\beta \right] \\ &\geq w_{min}^\beta \left[\min_{\Pi} \left(m_{\Pi(2)} + 2^\beta m_{\Pi(3)} + \cdots + (k-1)^\beta m_{\Pi(k)} \right) \right] \end{aligned}$$

which by Lemma 2 is equal to:

$$= w_{min}^\beta \left(m_2 + 2^\beta \cdot m_3 + \cdots + (k-1)^\beta \cdot m_k \right) \tag{20}$$

But,

$$\begin{aligned} C(\Pi_1) &= m_2 \cdot w_1^\beta + m_3 \cdot (w_1 + w_2)^\beta + \cdots + m_k \cdot (\sum_{i=1}^{k-1} w_i)^\beta. \\ &\leq w_{max}^\beta \left(m_2 + 2^\beta \cdot m_3 + \cdots + (k-1)^\beta m_k \right) \end{aligned} \tag{21}$$

From Equations 20 and 21, it follows that:

$$\frac{C(\Pi_1)}{C^*} \leq \frac{w_{max}^\beta}{w_{min}^\beta}.$$

This completes the proof of the theorem. □

6.2 The General Case

Having developed approximation algorithms for polynomial response functions, we next turn to the more general case of nondecreasing differentiable response functions. The maximization version of the single machine scheduling problem which uses concave functions is similar in structure to crackdown scheduling. It has been studied extensively and heuristics have been shown to be very close to the optimal (i.e, at worst 2/3 of the optimal). Unfortunately, the bounds developed do not apply to the corresponding minimization problem. We present instead two heuristics based on heuristics by Fisher–Krieger [11] and Alidaee [1] which have been shown empirically to perform well on the single machine scheduling problem.

6.2.1 Convex Response Functions

The input to this algorithm is a set of k drug markets $1, 2, \ldots, k$, and the output is a schedule Π. The i^{th} market has a *convex* response function f_i and its maintenance level is w_i.

This heuristic schedules crackdowns which are sequenced optimally relative to a linear approximation of the response functions. These linear approximations are specified at the time the market is considered for crackdown. Suppose several markets have been scheduled such that X resources are unavailable after the last market is suppressed. Then the linear approximation for any remaining market i is given by the linear function passing through the points $[X, -f_i(X)]$ and $[\sum_{i=1}^{k} w_i, -f(\sum_{i=1}^{k} w_i)]$. A market which maximizes $\frac{f_j(\sum_{\ell=1}^{k} w_\ell) - f_j(X)}{w_j}$ is chosen for crackdown. The formal algorithm follows.

> **Set** $X := 0$; $U := \{1, 2, \ldots, k\}$
>
> **for** $i = 0$ **to** k **do**
>
> > **begin**

Choose a (yet unscheduled) market j such that:

$$\frac{f_j\left(\sum_{\ell=1}^{k} w_\ell\right) - f_j(X)}{w_j} = \max_{\ell\ \in U}\left[\frac{f_\ell\left(\sum_{r=1}^{k} w_r\right) - f_\ell(X)}{w_\ell}\right].$$

{ If there are ties, choose the one with the smallest index }

Set $\Pi(i) := j$

Set $X := X + w_j$

Set $U := U - \{j\}$

end;

Output Π.

Fisher and Krieger have reported that this heuristic does extremely well empirically in the context of the single machine minimum tardiness problems, but for the CS problem, because of lack of data, its efficiency has not been substantiated through actual testing.

6.2.2 Differentiable Response Functions

We now present a heuristic which may be used for those instances of CS where the response function for each market is *differentiable* (and not necessarily convex). We assume that the input is a set of k markets with maintenance levels w_i and a differentiable nondecreasing response function f_i and the output is a schedule Π. The first derivative of f_i will be denoted by f_i'.

The heuristic schedules markets such that at any stage i, the market ℓ such that the ratio $\frac{f_\ell'(X\ +\ w_\ell\ [1/(k-i)])}{w_j}$ is largest among all remaining markets.

Set $X := 0$; $U := \{1, 2, \ldots, k\}$

for $i = 1$ **to** k **do**

  begin

    for $\ell \in U$ **do**

    begin

    Set $z_\ell := X + w_\ell \; [1/(k-i)] \cdot \sum w_r$

      {where the sum is taken over the maintenance levels of the remaining jobs in U.}

    end

  Choose a market j such that:

$$\frac{f_j'(z_j)}{w_j} = \max_{\ell \in U} \left[\frac{f_\ell'(z_\ell)}{w_\ell}\right]$$

    { If there are ties, choose the market with the smallest index }

  Set $\Pi(i) := j$; $X := X + w_j$

  Set $U := U - \{j\}$

  end. { of for loop }

Output Π.

Alidaee [1] has found empirically that this heuristic performs better than the one proposed by Fisher-Kreiger on the single machine scheduling problem, but again, its performance has not been evaluated in the context of drug markets.

Before closing this section, we present an alternative maximization model for the crackdown scheduling problem. The advantage of the maximization is that the tight theoretical bounds developed by Fisher–Krieger apply, but it is not as intuitive as the minimization version.

The maximization version of the crackdown scheduling problem is as follows. We are given a set of k markets $1, 2, \ldots, k$, such that market i has a maintenance level w_i. However, instead of a response function, we define a *benefit function* p_i corresponding to each market.

For market i, let t_i denote the time taken to bring down market i when $\sum_{j=1\ ,\ j\neq i}^{k} w_j$ resources are *unavailable*. In other words, t_i represents the time taken to suppress market i if it is scheduled last. If at any stage, the amount of unavailable resources is X, then let $f_i(X)$ denote the time taken to suppress market i. Define $p_i(X) = t_i - f_i(X)$ as the *benefit* which accrues if only X units of resources are unavailable, as opposed to $\sum_{j=1\ ,\ j\neq i}^{k} w_j$.

The objective is to find a schedule which maximizes benefits. That is, we need to compute P^* as given in the following expression.

$$P^* = \max_{\Pi} \left[\sum_{i=1}^{k} p_{\Pi(i)} \Big(\sum_{j\ =\ 1}^{i-1} w_{\Pi(j)} \Big) \right].$$

It can be shown that the schedule which maximizes this objective function also minimizes the objective function for the corresponding CS problem.

An approximation scheme analogous to the Fisher–Krieger algorithm given above yields solutions which are within 2/3 of the optimal solution.

7 Incorporating Displacement

As mentioned earlier, when there are multiple markets, drug dealers may respond to a crackdown by relocating to another market. This may be modeled by defining a more general response function.

In this case, the response function for market i is denoted by $f_i(X_1, X_2)$, where X_1 is the net amount of displacement from markets scheduled prior to market i, and X_2 is the amount of resources unavailable for cracking down on market i. We assume that f_i is monotonically increasing in X_1 and X_2. An instance of a crackdown scheduling problem with displacement will be referred as Crackdown Scheduling with Displacement (CSD).

Each drug market i is now characterized not just by its response function f_i and the maintenance level level w_i, but also by a displacement factor d_i. The displacement factor d_i denotes the amount of drug dealing displaced out of market i to other markets after a successful crackdown. We assume that if two markets i and j have the same displacement effect (i.e., $d_i = d_j$), then the net effect of displacement in any market l will be the same when market i is scheduled before market l, or when market j is scheduled before market l. Furthermore, we assume only "first–order" displacement. That is we assume that the second time a dealer is pushed out of a market he or she gives up street corner dealing, and, hence, is not displaced to any other market.

To illustrate, consider a set of three markets. If market 1 is scheduled first, then during the crackdown on the 2^{nd} and 3^{rd} markets, the displacement effect of market 1 on 2 and 3 is given by d_1. If market 2 is scheduled next followed by market 3, then the times taken to bring down markets 2 and 3 are given by: $f_2(d_1, w_1)$ and $f_3(d_1 + d_2, w_1 + w_2)$, respectively. Note that when market 1 is scheduled first, even though the displacement effect d_1 is the same for both markets 2 and 3, the impact of such a displacement is in general different for the two markets because of their different response functions.

Using the above logic, the general expression for the cost function can be written as

$$C(\Pi) = \left(\sum_{i=1}^{k} f_{\Pi(i)} (\sum_{j=1}^{i-1} d_{\Pi(j)} \ , \ \sum_{j=1}^{i-1} w_{\Pi(j)}) \right) .$$

Generalizing the response function in this manner makes this problem harder to solve. Consider the case in which for all i, $d_i = 0$. This is the same as the crackdown scheduling problem discussed above. Hence, the CSD is also NP-hard. Further, any instance of CS reduces to an instance of CSD with the special case that for all i, $w_i = 0$. This is done by identifying the displacement variables as maintenance levels. Hence even in this extremely restricted case, CSD is NP-hard.

A dynamic programming formulation of CSD is given by the following equations. Let S, M and C be as defined in Section 3.

$$\begin{aligned} \forall i \in \mathrm{M}\ ,\ C(\{i\}) &= f_i\Big(\sum_{\ell \in \mathrm{M}-\{i\}} d_\ell\ ,\ \sum_{j \in \mathrm{M}-\{i\}} w_j\Big) && (22)\\ C(S) &= \min_{j \in S}\left[f_j\left(\sum_{\ell \in \mathrm{M}-S} d_\ell\ ,\ \sum_{i \in \mathrm{M}-S} w_i\right) + C\left(S - \{j\}\right)\right] && (23) \end{aligned}$$

As for the crackdown scheduling problem, the optimal dynamic programming algorithm for CSD may not be practical when there are a large number of markets. Hence, as before, we consider some special cases where efficient algorithms may be obtained.

Case (i): The response functions $f_i(x_1, x_2)$ are all identical (i.e., for all i, $f_i \equiv f$) and the markets are such that $d_i \geq d_j \rightarrow w_i \geq w_j$. Then it can be shown that a strategy that gives priority to the market with the highest maintenance level yields an optimal solution. Note that the condition $d_i \geq d_j \rightarrow w_i \geq w_j$ is sufficient but not necessary for this result. Furthermore, under such an assumption, scheduling in nonincreasing order of the displacement factor is identical to scheduling in nonincreasing order of maintenance levels. The proof is similar to the proof of Theorem 3.

It is difficult to ascertain whether drug markets satisfy this condition. However, one can argue that the maintenance level required for a market might be an increasing function of the number of drug dealers left undisplaced in the market after the crackdown. This implies that the validity of the condition depends on what assumptions we make about the relative sizes of the markets. In the case when markets are of different size and dealer displacement d_i is proportional to the size of the market, the number of dealers left undisplaced is proportional to d_i. Hence, by our previous argument, the maintenance level w_i will be increasing in the displacement factor d_i, so this agreeability condition would hold. On the other hand, if the markets are all the same in size, then a contrary condition holds, namely, for all markets i and j, $w_i \geq w_j \rightarrow d_i \leq d_j$. Hence, it is difficult to say whether the condition holds in general.

Case (ii): The displacement factors are all identical, that is for all $i, d_i = d$, and the maintenance levels are also identical, that is for all $j, w_j = w$. Here, we can formulate the problem as an instance of the weighted bipartite matching problem, similar to the formulation in Section 5.3. The bipartite graph that is constructed for a given instance will be similar, except that now the weight of the edge (v_i, v_j) is $f_i((j-1)d, (j-1)w)$. An algorithm similar to the one in Section 5.3 will produce an optimal solution.

Case (iii): If the response functions are identical and either the maintenance levels or the displacement factors are the same for each market, then CSD reduces to the special case considered in Section 5.2.

8 Summary

This paper addresses the important question of how to schedule crackdowns on drug markets. Our formulation was predicated on two previous results: 1) that it is optimal to apply the maximum available effort to the target drug market, and 2) that a crackdown needs to be followed by a maintenance phase to preserve the gains achieved.

After showing that the general problem of Crackdown Scheduling is NP-complete, we presented a dynamic programming formulation and exponential time algorithm for finding an optimal ordering of drug markets. Then we considered the following special cases of the general problem.

- The onus of maintaining the gains achieved by a crackdown is not on the group conducting the crackdown, e.g. it is the responsibility of uniformed patrol, volunteer groups, or other government agencies;
- the maintenance level required in each market is the same;
- the response function in each market is the same; and
- the response function for each market is linear in the amount of resources available.

Besides giving optimal solutions to the above special cases, we give approximation algorithms for the more general cases of polynomial response functions, and heuristics for convex and differentiable functions. The optimal or close-to-optimal solutions for all these cases give priority to markets that take longer to bring down and those that require fewer resources to maintain. We then extended some of these results to the situation in which dealers can be displaced from one market to another after a crackdown. This examination suggested adding another rule of thumb: give priority to markets from which the least number of dealers will relocate to the remaining markets.

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Appendix

A Polynomial Response Functions

This appendix gives an efficient algorithm for solving the Crackdown Scheduling problem when the response functions of the markets can be modeled as polynomials of the form $f_i(X) = m_i \cdot X^\beta + c$, for a constant integer $\beta > 1$.

The optimal algorithm assumes *agreeability* among markets, which implies that for all markets i, j, $w_i < w_j \rightarrow m_i > m_j$. Such an assumption implies that if the maintenance level for market i is less that the maintenance level for market j, then market i will take longer to bring down. We assume without loss of generality that all maintenance levels are positive numbers greater than 1.

We will now show that a strategy that schedules the markets in nonincreasing order of the ratio $\frac{m_i}{w_i}$ produces an optimal solution.

Theorem 8 *Let* $\{1, 2, \ldots, k\}$ *be a given set of agreeable markets with maintenance levels* $\{w_i\}$ *and response functions* $f_i(X) = m_i X^\beta + c_i$*. Then a schedule* Π *with the property that*

$$\frac{m_{\Pi(1)}}{w_{\Pi(1)}} \geq \frac{m_{\Pi(2)}}{w_{\Pi(2)}} \geq \cdots \frac{m_{\Pi(n)}}{w_{\Pi(n)}}$$

is an optimal solution.

Proof Let Π be a schedule with the property given above. Suppose there exists an optimal schedule Π_1 such that for some i, $\frac{m_{\Pi_1(i)}}{w_{\Pi_1(i)}} < \frac{m_{\Pi_1(i+1)}}{w_{\Pi_1(i+1)}}$. Consider the schedule $\overline{\Pi_1}$ which is identical to Π_1 except that it interchanges the order of markets $\Pi_1(i)$ and $\Pi_1(i+1)$.

Then, on substituting values of f_i into Equation 2, we have:

$$C(\Pi_1) \; - \; C(\overline{\Pi_1}) \; = \; \sum_{j=0}^{\beta} m_{\Pi_1(i+1)} \cdot w^j_{\Pi_1(i)} \; - \; m_{\Pi_1(i)} \cdot w^j_{\Pi_1(i+1)}. \tag{24}$$

Since $\frac{m_{\Pi_1(i+1)}}{w_{\Pi_1(i+1)}} \; > \; \frac{m_{\Pi_1(i)}}{w_{\Pi_1(i)}}$ and since $w_{\Pi_1(i)}, w_{\Pi_1(i+1)} \geq 1$, $C(\Pi_1) - C(\overline{\Pi_1}) \; > \; 0$, which is a contradiction. This completes the proof. $\square$

Note that the agreeability condition is a sufficient but not a necessary condition for this scheduling strategy to be optimal. Given the primitive state of understanding about drug markets, it is difficult to judge how often it would be satisfied in practice.