A More Practical Algorithm for Drawing Binary Trees in Linear Area with Arbitrary Aspect Ratio*

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Abstract

Trees are usually drawn planar, i.e. without any edge-crossings. It is important to minimize the area of a drawing, so that it can fit within the given drawing-window. It is also important to give user some control over the aspect ratio of the drawing, so that she can display the drawing in a drawing-window with arbitrary width-to-height ratio. In a grid drawing, each node is assigned integer coordinates. In [6], it was shown that any binary tree with n nodes admits a planar straight-line grid drawing with area $O(n)$ and with any pre-specified aspect ratio in the range $[n^{-\epsilon}, n^{\epsilon}]$, where ϵ is any constant such that $0 < \epsilon < 1$. It was also shown that such a drawing can be constructed in $O(n \log n)$ time. In particular, this showed that optimal area (equal to $O(n)$) and optimal aspect ratio (equal to 1) is simultaneously achievable for such drawings.

However, the algorithm of [6] is not suitable for practical use. The main problem is that the constant c hidden in the "Oh" notation for area is quite large (for example, it can be as high as 3900).

In this paper, we make several non-trivial practical improvements to the algorithm, which make it suitable for practical use. We have also conducted experiments on this newer version of the algorithm for randomly-generated binary trees with up to 50,000 nodes, and for complete binary trees with up to $65,535 = 2^{16} - 1$ nodes. Our experiments show that it constructs area-efficient drawings in practice, with area at most 8 times the number of nodes for complete binary trees, and at most 10 times the number of nodes for randomly-generated binary trees.

1 Introduction

Trees are very common data-structures, which are used to model information in a variety of applications. A *drawing* Γ of a tree T maps each node of T to a distinct point in the plane, and each edge (u, v) of T to a simple Jordan curve with endpoints u and v. Γ is a *straight-line* drawing, if each edge is drawn as a single line-segment. Γ is a *grid* drawing if all the nodes have integer coordinates. Γ is a *planar* drawing, if edges do not intersect each other in the drawing. In this paper, we concentrate on grid drawings. So, we will assume that the plane is covered by a rectangular grid. Let R be a rectangle with sides parallel to the X- and Y-axes. The *width* (*height*) of R is equal to the number of grid points with the same y (x) coordinate contained within R. The *area* of R is equal to the number of grid points contained within R. The *aspect ratio* of R is the ratio of its width and height. R is the *enclosing rectangle* of Γ, if it is the smallest rectangle that covers the entire drawing. The *width*, *height*, *area*, and *aspect ratio* of Γ is equal to the width, height, area, and aspect ratio, respectively, of its enclosing rectangle. T is a binary tree if each node has at most two children.

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2 Our Result

Planar straight-line drawings are more aesthetically pleasing than non-planar polyline drawings. Grid drawings guarantee at least unit distance separation between the nodes of the tree, and the integer coordinates of the nodes and edge-bends allow the drawings to be displayed in a display surface, such as a computer screen, without any distortions due to truncation and rounding-off errors. Giving users control over the aspect ratio of a drawing allows them to display the drawing in different kinds of display surfaces with different aspect ratios. Finally, it is important to minimize the area of a drawing, so that the users can display a tree in as small drawing area as possible.

It is therefore important to investigate the problem of constructing planar straight-line grid drawings of binary trees with small area. Clearly, any planar grid drawing of a binary tree with n nodes requires $\Omega(n)$ area. A long-standing fundamental problem, therefore, was that whether this is a tight bound also, i.e., given a binary tree T with n nodes, can we construct a planar straight-line grid drawing of T with area $O(n)$?

[6] settled this problem by proving that a binary tree can be drawn in linear area with arbitrary aspect ratio in the range $[n^{-\epsilon}, n^{\epsilon}]$, where ϵ is any constant, such that $0 < \epsilon < 1$, in $O(n \log n)$ time. In particular, this result showed that optimal area (equal to $O(n)$) and optimal aspect ratio (equal to 1) is simultaneously achievable (see Corollary 1).

While the result of [6] was significant from a theoretical point of view, the drawing-algorithm presented in [6] suffered from the following drawbacks, that made it unsuitable for practical use:

- The constant c hidden in the "Oh" notation for area can be quite large (for example, it can be as high as 3900 for $\epsilon = 0.6$, and $A = 1$). One might argue that c is really a worst-case bound, and the algorithm might perform better in practice. However, the problem is that given a tree T with n nodes, and two numbers ϵ and A as input, the algorithm will always pre-allocate a rectangle R with size *exactly equal* to cn, and draw T within R. Thus, the area of R is always equal to the worst-case area, and correspondingly, the drawing also has a large area. This is the major drawback of this algorithm.
- Also, it uses another algorithm, called *Algorithm u*-HV-Draw*, as a subroutine. This increases the complexity of implementing the algorithm.

In this paper, we have made several practical improvements to the algorithm, which make it more suitable for practical use: (Note that the area of the drawing constructed by this newer version of the algorithm is still $O(n)$)

- We have developed a newer version of the algorithm that does not require the pre-assignment of a rectangle with the worst-case area to draw a tree. This makes it possible for the algorithm to construct a more area-efficient drawing of a tree in practice than that by the algorithm of [6].
  The main difference between this newer version and the algorithm of [6] is as follows: The algorithm of [6] is a recursive algorithm, which, in each recursive step, splits a tree T into several small trees, correspondingly splits its pre-assigned rectangle R into several smaller rectangles, assigns these smaller rectangles to the smaller trees, recursively draws the smaller trees within their pre-assigned rectangles, and then combine their drawings to obtain a drawing of T. In this newer version, instead of pre-assigning a rectangle, we only pre-assign an aspect ratio to each smaller tree, recursively construct a drawing of each smaller tree using its pre-assigned aspect ratio, and then combine their drawings to obtain a drawing of T.
- This newer version does not require *Algorithm u*-HV-Draw* as a subroutine, which makes it easier to implement.
- The proof for the area (see Lemma 4) given in [6] is based on a theorem by Leslie Valiant (Theorem 6 of [10]). Unfortunately, the most natural way of using the theorem seemed to be requiring the pre-assignment of a rectangle (with a large area). Hence, developing an algorithm that does not pre-assign a rectangle required developing a new proof. Correspondingly, we have developed a new proof that does not use the theorem. Instead, it simply uses mathematical induction.
- We have also implemented this newer version, and experimentally evaluated its performance for randomly-generated binary trees with up to 50,000 nodes, and for complete binary trees with up to $65,535 = 2^{16} - 1$ nodes. Our experiments show that it constructs area-efficient drawings in practice, with area at most at most 8 times the number of nodes for complete binary trees, and at most 10 times

the number of nodes for randomly-generated binary trees.

3 Previous Results

Previous to the result of [6], the best-known upper bound on the area of a planar straight-line grid drawing of an n-node binary tree was $O(n \log\log n)$, which was shown in [1] and [8].

However, note that the $O(n \log\log n)$ area drawing constructed by the algorithms of [1] and [8] has a fixed aspect ratio, equal to $\theta(\log^2 n/(n \log\log n))$, whereas the aspect ratio of the drawing constructed by our algorithm can be specified by the user.

We now summarize some other known results on planar grid drawings of binary trees (for more results, see [4]). Let T be an n-node binary tree. [5] presents an algorithm for constructing an upward polyline drawing of T with $O(n)$ area, and any user-specified aspect ratio in the range $[n^{-\epsilon}, n^{\epsilon}]$, where ϵ is any constant, such that $0 < \epsilon < 1$. [7] and [10] present algorithms for constructing a (non-upward) orthogonal polyline drawing of T with $O(n)$ area. [1] gives an algorithm for constructing an upward orthogonal straight-line drawing of T with $O(n \log n)$ area, and any user-specified aspect ratio in the range $[\log n/n, n/\log n]$. It also shows that $n \log n$ is also a tight bound for such drawings. [8] gives an algorithm for constructing an upward straight-line drawing of T with $O(n \log\log n)$ area. If T is a Fibonacci tree, (AVL tree, complete binary tree), then [2, 9] ([3], [2], respectively) give algorithms for constructing an upward straight-line drawing of T with $O(n)$ area.

Table 1 summarizes these results.

Tree Type	Drawing Type	Area	Aspect Ratio	Reference
Fibonacci	Upward Straight-line	$O(n)$	$\theta(1)$	[2, 9]
AVL	Upward Straight-line	$O(n)$	$\theta(1)$	[3]
Complete Binary	Upward Straight-line	$O(n)$	$\theta(1)$	[2]
General Binary	Upward Orthogonal Polyline	$O(n \log\log n)$	$\theta(\log^2 n/(n \log\log n))$	[5, 8]
	(Non-upward) Orthogonal Polyline	$O(n)$	$\theta(1)$	[7, 10]
	Upward Orthogonal Straight-line	$O(n \log n)$	$[\log n/n, n/\log n]$	[1]
	Upward Polyline	$O(n)$	$[n^{-\epsilon}, n^{\epsilon}]$	[5]
	Upward Straight-line	$O(n \log\log n)$	$\theta(\log^2 n/(n \log\log n))$	[8]
	(Non-upward) Straight-line	$O(n \log\log n)$	$\theta(\log^2 n/(n \log\log n))$	[1]
		$O(n)$	$[n^{-\epsilon}, n^{\epsilon}]$	[6] and *this paper*

Table 1: Bounds on the areas and aspect ratios of various kinds of planar grid drawings of an n-node binary tree. Here, ϵ is an arbitrary constant, such that $0 < \epsilon < 1$.

4 Preliminaries

Throughout the paper, by the term *tree*, we will mean a rooted tree, i.e., a tree with a given root. We will assume that the plane is covered by an infinite rectangular grid. A *horizontal channel* (*vertical channel*) is a line parallel to the X-(Y-)axis, passing through grid-points.

Let T be a tree with root o. Let n be the number of nodes in T. T is an *ordered* tree if the children of each node are assigned a left-to-right order. A *partial tree* of T is a connected subgraph of T. If T is an ordered tree, then the *leftmost path* p of T is the maximal path consisting of nodes that are leftmost children, except the first one, which is the root of T. The last node of p is called the *leftmost* node of T. Two nodes of T are *siblings* if they have the same parent. The *subtree* of T rooted at a node v consists of v and all the descendents of v. T is the *empty tree*, i.e., $T = \emptyset$, if it has zero nodes in it.

Let Γ be a drawing of T. By the *bottom* (*top*, *left*, and *right*, respectively) boundary of Γ, we will mean the *bottom* (*top*, *left*, and *right*, respectively) boundary of the enclosing rectangle $R(\Gamma)$ of Γ. Similarly, by *top-left* (*top-right*, *bottom-left*, and *bottom-right*, respectively) corner of Γ, we mean the *top-left* (*top-right*, *bottom-left*, and *bottom-right*, respectively) corner of $R(\Gamma)$.

Let R be a rectangle, such that Γ is entirely contained within R. R has a *good* aspect ratio, if its aspect ratio is in the range $[n^{-\epsilon}, n^{\epsilon}]$, where $0 < \epsilon < 1$ is a constant.

Let w be a node of an ordered tree. We denote by $p(w)$, $l(w)$, $r(w)$, and $s(w)$, respectively, the parent, left child, right child, and sibling of w.

For some trees, we will designate a special *link* node u^*, that has at most one child. As we will see later in Section 5, the link node helps in combining the drawing of a tree with the drawing of another tree to obtain a drawing of a larger tree, that contains both the trees.

Let T be a tree with link node u^*. Let o be the root of T. A planar straight-line grid drawing Γ of T is a *feasible* drawing of T, if it has the following three properties:

- **Property 1**: The root o is placed at the top-left corner of Γ.
- **Property 2**: If $u^* \neq o$, then u^* is placed at the bottom boundary of Γ. Moreover, we can move u^* downwards in its vertical channel by any distance without causing any edge-crossings in Γ.
- **Property 3**: If $u^* = o$, then no other node or edge of T is placed on, or crosses the vertical and horizontal channels occupied by o. Moreover, we can move u^* (i.e., o) upwards in its vertical channel by any distance without causing any edge-crossings in Γ.

The following Theorem paraphrases a theorem of Valiant [10]:

Theorem 1 (Separator Theorem [10]) *Every binary tree T with n nodes, where $n \geq 2$, contains an edge e, called a* separator edge*, such that removing e from T splits it into two non-empty trees with n_1 and n_2 nodes, respectively, such that for some x, where $1/3 \leq x \leq 2/3$, $n_1 = xn$, and $n_2 = (1-x)n$. Moreover, e can be found in $O(n)$ time.*

Let Γ be a drawing of T. Let v be a node of T located at grid point (i, j) in Γ. Assume that the root o of T is located at the grid point $(0, 0)$ in Γ. We define the following operations on Γ (see Figure 1):

- *rotate operation*: rotate Γ counterclockwise by δ degrees around o. After a rotation by δ degrees of Γ, node v will get relocated to the point $(i\cos\delta - j\sin\delta, i\sin\delta + j\cos\delta)$. In particular, after rotating Γ by 90°, node v will get relocated to the grid point $(-j, i)$.
- *flip operation*: flip Γ vertically or horizontally about the $X-$ or $Y-$axis, respectively. After a horizontal flip of Γ, node v will be located at grid point $(-i, j)$. After a vertical flip of Γ, node v will be located at grid point $(i, -j)$.

Suppose Γ were a feasible drawing, where the link node u^* was placed at the bottom of Γ. On applying a rotation operation followed by a vertical-flip operation, u^* will get relocated to the right-boundary of Γ, but o will continue to stay at the top-left corner (see Figure 1). We will use this fact later in Section 5 in the *Compose Drawings* step of our drawing algorithm.

5 Our Tree Drawing Algorithm

Let T be a binary tree with link node u^*. Let n be the number of nodes in T. Let A and ϵ be two numbers, where ϵ is a constant, such that $0 < \epsilon < 1$, and $n^{-\epsilon} \leq A \leq n^{\epsilon}$. A is called the *desirable aspect ratio* for T.

Our tree drawing algorithm, called *DrawTree*, takes ϵ, A, and T as input, and uses a simple divide-and-conquer strategy to recursively construct a feasible drawing Γ of T, by performing the following actions at each recursive step:

- *Split Tree*: Split T into at most five partial trees by removing at most two nodes and their incident edges from it. Each partial tree has at most $(2/3)n$ nodes. Based on the arrangement of these partial trees within T, we get two cases, which are shown in Figures 3 and 4, respectively, and described later in Section 5.1.
- *Assign Aspect Ratios*: Correspondingly, assign a desirable aspect ratio A_k to each partial tree T_k. The value of A_k is based on the value of A and the number of nodes in T_k.

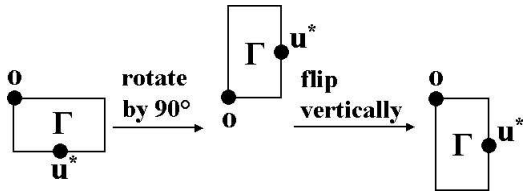


Figure 1: Rotating a drawing Γ by $90°$, followed by flipping it vertically. Note that initially node u^* was located at the bottom boundary of Γ, but after the rotate operation, u^* is at the right boundary of Γ.

- *Draw Partial Trees*: Recursively construct a feasible drawing of each partial tree T_k with A_k as its desirable aspect ratio.
- *Compose Drawings*: Arrange the drawings of the partial trees, and draw the nodes and edges, that were removed from T to split it, such that the drawing Γ of T is a feasible drawing. Note that the arrangement of these drawings is done based on the cases shown in Figures 3 and 4. In each case, if $A < 1$, then the drawings of the partial trees are stacked one above the other, and if $A \geq 1$, then they are placed side-by-side.

Remark: The drawing Γ constructed by the algorithm may not have aspect ratio exactly equal to A, but as we will prove later in Lemma 4, it will fit inside a rectangle with area $O(n)$ and aspect ratio A.

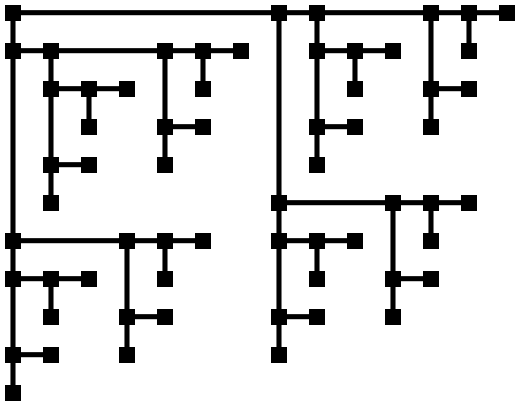


Figure 2: Drawing of the complete binary tree with 63 nodes constructed by Algorithm *DrawTree*, with $A = 1$ and $\epsilon = 0.5$.

Figure 2 shows a drawing of a complete binary tree with 63 nodes constructed by Algorithm *DrawTree*, with $A = 1$ and $\epsilon = 0.5$.

We now give the details of each action performed by Algorithm *DrawTree*:

5.1 Split Tree

The splitting of tree T into partial trees is done as follows:

- Order the children of each node such that u^* becomes the leftmost node of T.
- Using Theorem 1, find a separator edge (u, v) of T, where u is the parent of v.
- Based on whether, or not, (u, v) is in the leftmost path of T, we get two cases:
  - *Case 1: The separator edge (u, v) is not in the leftmost path of T.* Let o be the root of T. Let a

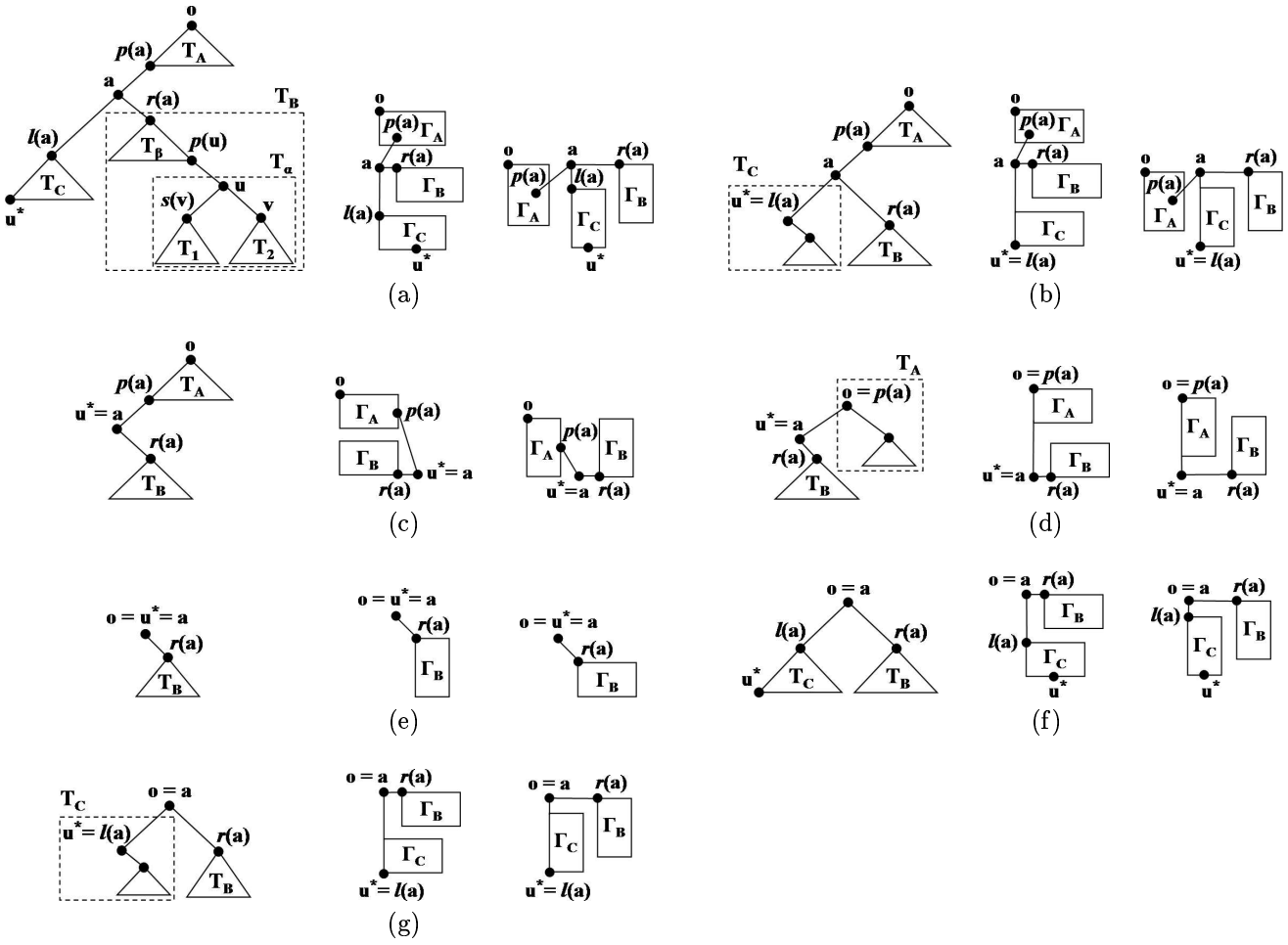


Figure 3: Drawing T in all the seven subcases of Case 1 (where the separator edge (u, v) is not in the leftmost path of T): (a) $T_A \neq \emptyset$, $T_C \neq \emptyset$, $u^* \neq l(a)$, (b) $T_A \neq \emptyset$, $T_C \neq \emptyset$, $u^* = l(a)$, (c) $T_A \neq \emptyset$, $T_C = \emptyset$, $o \neq p(a)$, (d) $T_A \neq \emptyset$, $T_C = \emptyset$, $o = p(a)$, (e) $T_A = \emptyset$, $T_C = \emptyset$, (f) $T_A = \emptyset$, $T_C \neq \emptyset$, $u^* \neq l(a)$, and (g) $T_A = \emptyset$, $T_C \neq \emptyset$, $u^* = l(a)$. For each subcase, we first show the structure of T for that subcase, then its drawing when $A < 1$, and then its drawing when $A \geq 1$. In Subcases (a) and (b), for simplicity, $p(a)$ is shown to be in the interior of Γ_A, but actually, either it is the same as o, or if $A < 1$ ($A \geq 1$), then it is placed at the bottom (right) boundary of Γ_A. For simplicity, we have shown Γ_A, Γ_B, and Γ_C as identically sized boxes, but in actuality, they may have different sizes.

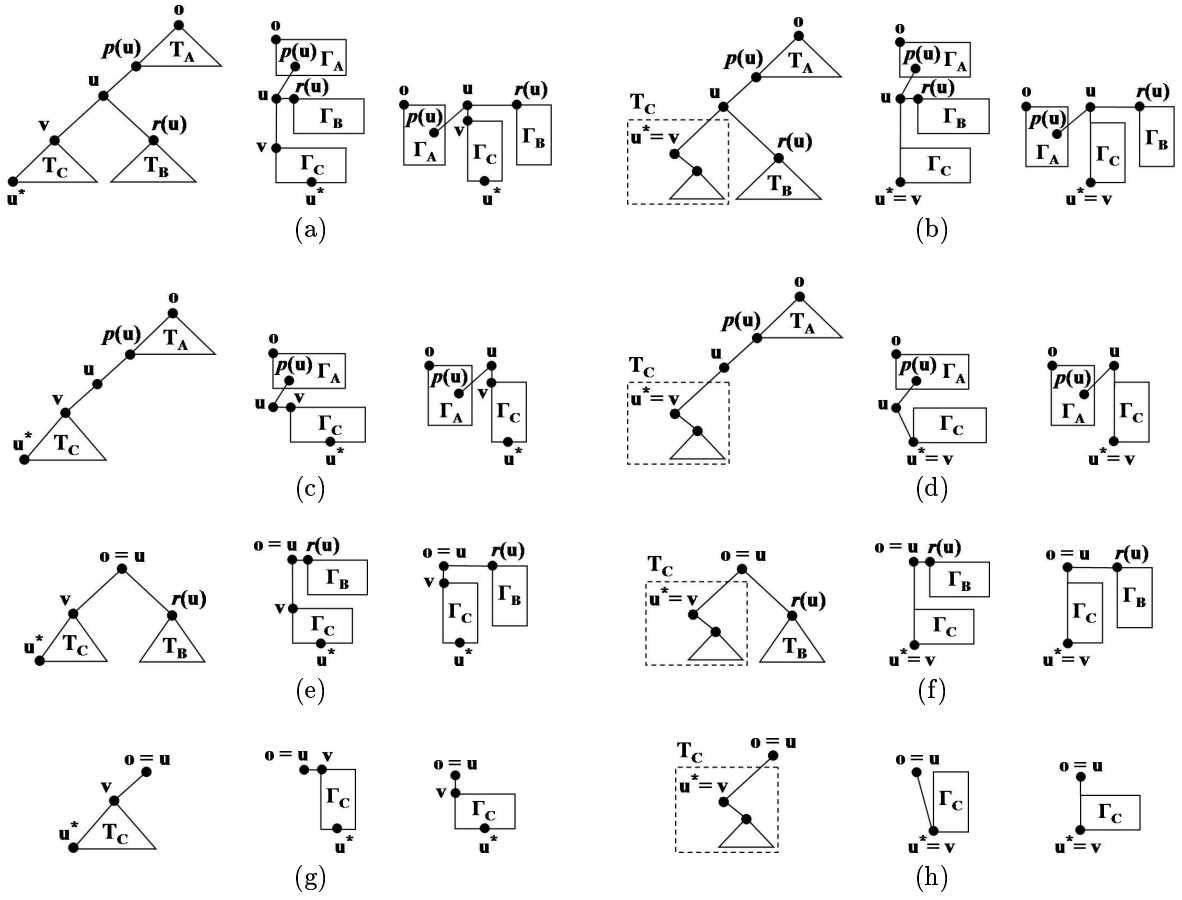


Figure 4: Drawing T in all the eight subcases of Case 2 (where the separator edge (u, v) is in the leftmost path of T): (a) $T_A \neq \emptyset$, $T_B \neq \emptyset$, $v \neq u^*$, (b) $T_A \neq \emptyset$, $T_B \neq \emptyset$, $v = u^*$, (c) $T_A \neq \emptyset$, $T_B = \emptyset$, $v \neq u^*$, (d) $T_A \neq \emptyset$, $T_B = \emptyset$, $v = u^*$, (e) $T_A = \emptyset$, $T_B \neq \emptyset$, $v \neq u^*$, (f) $T_A = \emptyset$, $T_B \neq \emptyset$, $v = u^*$, (g) $T_A = \emptyset$, $T_B = \emptyset$, $v \neq u^*$, and (h) $T_A = \emptyset$, $T_B = \emptyset$, $v = u^*$. For each subcase, we first show the structure of T for that subcase, then its drawing when $A < 1$, and then its drawing when $A \geq 1$. In Subcases (a), (b), (c), and (d), for simplicity, $p(u)$ is shown to be in the interior of Γ_A, but actually, either it is same as o, or if $A < 1$ ($A \geq 1$), then it is placed at the bottom (right) boundary of Γ_A. For simplicity, we have shown Γ_A, Γ_B, and Γ_C as identically sized boxes, but in actuality, they may have different sizes.

be the last node common to the path $o \rightsquigarrow v$, and the leftmost path of T. We define partial trees T_A, T_B, T_C, T_α, T_β, T_1 and T_2, as follows (see Figure 3(a)):

* If $o \neq a$, then T_A is the maximal partial tree with root o, that contains $p(a)$, but does not contain a. If $o = a$, then $T_A = \emptyset$.
* T_B is the subtree rooted at $r(a)$.
* If $u^* \neq a$, then T_C is the subtree rooted at $l(a)$. If $u^* = a$, then $T_C = \emptyset$.
* If $s(v)$ exists, i.e., if v has a sibling, then T_1 is the subtree rooted at $s(v)$. If v does not have a sibling, then $T_1 = \emptyset$.
* T_2 is the subtree rooted at v.
* If $u \neq a$, then T_α is the subtree rooted at u. If $u = a$, then $T_\alpha = T_2$. Note that T_α is a subtree of T_B.
* If $u \neq a$ and $u \neq r(a)$, then T_β is the maximal partial tree with root $r(a)$, that contains $p(u)$, but does not contain u. If $u = a$ or $u = r(a)$, then $T_\beta = \emptyset$. Again, note that T_β belongs to T_B.

We get seven subcases, where subcase (a) is the general case, and subcases (b–g) are special cases: (a) $T_A \neq \emptyset$, $T_C \neq \emptyset$, $u^* \neq l(a)$ (see Figure 3(a)), (b) $T_A \neq \emptyset$, $T_C \neq \emptyset$, $u^* = l(a)$ (see Figure 3(b)), (c) $T_A \neq \emptyset$, $T_C = \emptyset$, $o \neq p(a)$ (see Figure 3(c)), (d) $T_A \neq \emptyset$, $T_C = \emptyset$, $o = p(a)$ (see Figure 3(d)), (e) $T_A = \emptyset$, $T_C = \emptyset$ (see Figure 3(e)), (f) $T_A = \emptyset$, $T_C \neq \emptyset$, $u^* \neq l(a)$ (see Figure 3(f)), and (g) $T_A = \emptyset$, $T_C \neq \emptyset$, $u^* = l(a)$ (see Figure 3(g)).

The reason we get these seven subcases is as follows: T_2 has at least $n/3$ nodes in it because of Theorem 1. Hence $T_2 \neq \emptyset$, and so, $T_B \neq \emptyset$. Based on whether $T_A = \emptyset$ or not, $T_C = \emptyset$ or not, $u^* = l(a)$ or not, and $o = p(a)$ or not, we get a total of sixteen cases. From these sixteen cases, we obtain the above seven subcases, by grouping some of them together. For example, the cases $T_A \neq \emptyset$, $T_C \neq \emptyset$, $u^* \neq l(a)$, $o = p(a)$, and $T_A \neq \emptyset$, $T_C \neq \emptyset$, $u^* \neq l(a)$, $o \neq p(a)$ are grouped together to give Case (a), i.e., $T_A \neq \emptyset$, $T_C \neq \emptyset$, $u^* \neq l(a)$. So, Case (a) corresponds to 2 cases. Similarly, Cases (b), (c), (d), (f), and (g) correspond to 2 cases each, and Case (e) corresponds to 4 cases.

In each case, we remove nodes a and u (which could be the same node as a), and their incident edges, to split T into at most five partial trees T_A, T_C, T_β, T_1, and T_2. We also designate $p(a)$ as the link node of T_A, $p(u)$ as the link node of T_β, and u^* as the link node of T_C. We arbitrarily select a node of T_1 that has at most one child (for example, we can select a leaf), and designate it as the link node of T_1. We arbitrarily select a node of T_2 that has at most one child (for example, we can select a leaf), and designate it as the link node of T_2.

– *Case 2: The separator edge (u, v) is in the leftmost path of T.* Let o be the root of T. We can define partial trees T_A, T_B, and T_C as follows (see Figure 4(a)):

* If $o \neq u$, then T_A is the maximal partial tree with root o, that contains $p(u)$, but does not contain u. If $o = u$, then $T_A = \emptyset$.
* If $r(u)$ exits, i.e., u has a right child, then T_B is the subtree rooted at $r(u)$. If u does not have a right child, then $T_B = \emptyset$.
* T_C is the subtree rooted at v.

We get eight subcases, where subcase (a) is the general case, and subcases (b–h) are special cases: (a) $T_A \neq \emptyset$, $T_B \neq \emptyset$, $v \neq u^*$ (see Figure 4(a)), (b) $T_A \neq \emptyset$, $T_B \neq \emptyset$, $v = u^*$ (see Figure 4(b)), (c) $T_A \neq \emptyset$, $T_B = \emptyset$, $v \neq u^*$ (see Figure 4(c)), (d) $T_A \neq \emptyset$, $T_B = \emptyset$, $v = u^*$ (see Figure 4(d)), (e) $T_A = \emptyset$, $T_B \neq \emptyset$, $v \neq u^*$ (see Figure 4(e)), (f) $T_A = \emptyset$, $T_B \neq \emptyset$, $v = u^*$ (see Figure 4(f)), (g) $T_A = \emptyset$, $T_B = \emptyset$, $v \neq u^*$ (see Figure 4(g)), and (h) $T_A = \emptyset$, $T_B = \emptyset$, $v = u^*$ (see Figure 4(h)).

The reason we get these eight subcases is as follows: T_C has at least $n/3$ nodes in it because of Theorem 1. Hence, $T_C \neq \emptyset$. Based on whether $T_A = \emptyset$ or not, $T_B = \emptyset$ or not, and $v = u^*$ or not, we get the eight subcases given above.

In each case, we remove node u, and its incident edges, to split T into at most three partial trees T_A, T_B, and T_C. We also designate $p(u)$ as the link node of T_A, and u^* as the link node of T_C. We arbitrarily select a node of T_B that has at most one child (for example, we can select a leaf), and designate it as the link node of T_B.

Remark: In Case 2, from the definition of the separator edge (u, v) (see Theorem 1), it can be easily shown that $T_A = \emptyset$ and $T_B = \emptyset$ can happen simultaneously only if T has very few nodes in it, namely, at most 5 nodes. Hence, Case 2(g) and Case 2(h) can occur only if T has at most 5 nodes in it.

5.2 Assign Aspect Ratios

Let T_k be a partial tree of T, where for Case 1, T_k is either T_A, T_C, T_β, T_1, or T_2, and for Case 2, T_k is either T_A, T_B, or T_C. Let n_k be number of nodes in T_k.

Definition: T_k is a *large* partial tree of T if:

- $A \geq 1$ and $n_k \geq (n/A)^{1/(1+\epsilon)}$, or
- $A < 1$ and $n_k \geq (An)^{1/(1+\epsilon)}$,

and is a *small* partial tree of T otherwise.

In Step *Assign Aspect Ratios*, we assign a desirable aspect ratio A_k to each non-empty T_k as follows: Let $x_k = n_k/n$.

- If $A \geq 1$: If T_k is a large partial tree of T, then $A_k = x_k A$, otherwise (i.e., if T_k is a small partial tree of T) $A_k = n_k^{-\epsilon}$.
- If $A < 1$: If T_k is a large partial tree of T, then $A_k = A/x_k$, otherwise (i.e., if T_k is a small partial tree of T) $A_k = n_k^{\epsilon}$.

Intuitively, the above assignment strategy ensures that each partial tree gets a good desirable aspect ratio.

5.3 Draw Partial Trees

First, we change the values of A_A and A_β in some situations, as follows: (recall that A_A and A_β are the desirable aspect ratios for T_A and T_β, respectively, when they are non-empty trees)

- In Case 1(c), we change the value of A_A to $1/A_A$. Moreover, in Case 1(c), if $A \geq 1$, then we change the value of A_β also to $1/A_\beta$.
- In Cases 1(a) and 1(b), if $A \geq 1$, then we change the values of A_A and A_β to $1/A_A$ and $1/A_\beta$, respectively.
- In Cases 1(d), 1(e), 1(f), and 1(g), if $A \geq 1$, then we change the values of A_β to $1/A_\beta$.
- In Cases 2(a), 2(b), 2(c), and 2(d), if $A \geq 1$, then we change the value of A_A to $1/A_A$.

(This is done so because later in Step *Compose Drawings*, when constructing Γ,

- in Case 1(c), the drawing of T_A is rotated by 90°, and if $A \geq 1$, then the drawing of T_β is also rotated by 90°,
- in Cases 1(a) and 1(b), if $A \geq 1$, then the drawings of T_A and T_β are rotated by 90°,
- in Cases 1(d), 1(e), 1(f), and 1(g), if $A \geq 1$, then the drawing of T_β is rotated by 90°, and
- in Cases 2(a), 2(b), 2(c), and 2(d), if $A \geq 1$, then the drawing of T_A is rotated by 90°.

Drawing T_A and T_β with desirable aspect ratios $1/A_A$ and $1/A_\beta$, respectively, compensates for the rotation, and ensures that the drawings of T_A and T_β that eventually get placed within Γ are those with desirable aspect ratios A_A and A_β, respectively.)

Next, we draw recursively each non-empty partial tree T_k with A_k as its desirable aspect ratio. The base case for the recursion happens when T_k contains exactly one node, in which case, the drawing of T_k is simply the one consisting of exactly one node.

5.4 Compose Drawings

Let Γ_k denote the drawing of a partial tree T_k constructed in Step *Draw Partial Trees*. We now describe the construction of a feasible drawing Γ of T from the drawings of its partial trees in both Cases 1 and 2.

In Case 1, we first construct a drawing Γ_α of the partial tree T_α by composing Γ_1 and Γ_2 as shown in Figure 5, then construct a drawing Γ_B of T_B by composing Γ_α and Γ_β as shown in Figure 6, and finally construct Γ by composing Γ_A, Γ_B and Γ_C as shown in Figure 3.

Γ_α is constructed as follows (see Figure 5): (Recall that if $u \neq a$ then T_α is the subtree of T rooted at u, otherwise $T_\alpha = T_2$)

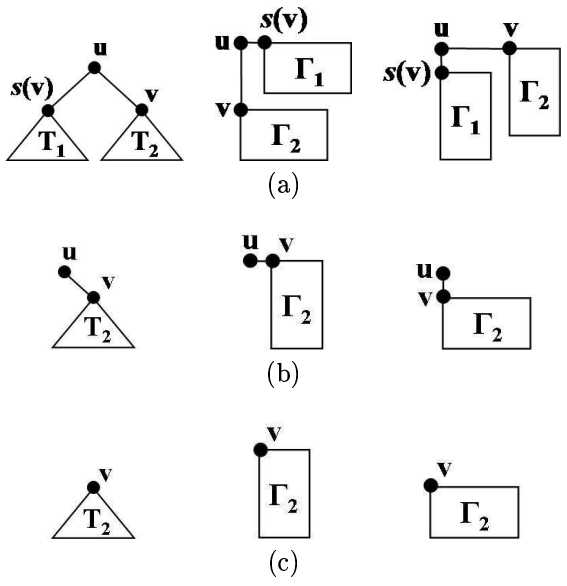


Figure 5: Drawing T_α, when: (a) $u \neq a$ and $T_1 \neq \emptyset$, (b) $u \neq a$ and $T_1 = \emptyset$, and (c) $u = a$. For each case, we first show the structure of T_α for that case, then its drawing when $A < 1$, and then its drawing when $A \geq 1$. For simplicity, we have shown Γ_1 and Γ_2 as identically sized boxes, but in actuality, their sizes may be different.

- If $u \neq a$ and $T_1 \neq \emptyset$ (see Figure 5(a)), then:
  - If $A < 1$, then place Γ_1 above Γ_2 such that the left boundary of Γ_1 is one unit to the right of the left boundary of Γ_2. Place u in the same vertical channel as v and in the same horizontal channel as $s(v)$.
  - If $A \geq 1$, then place Γ_1 one unit to the left of Γ_2, such that the top boundary of Γ_1 is one unit below the top boundary of Γ_2. Place u in the same vertical channel as $s(v)$ and in the same horizontal channel as v.

  Draw edges $(u, s(v))$ and (u, v).
- If $u \neq a$ and $T_1 = \emptyset$ (see Figure 5(b)), then:
  - If $A < 1$, then place u one unit to the left of Γ_2 in the same horizontal channel as v.
  - If $A \geq 1$, then place u one unit above Γ_2 in the same vertical channel as v.

  Draw edge (u, v).
- If $u = a$, then Γ_α is the same as Γ_2 (see Figure 5(c)).

Γ_B is constructed as follows (see Figure 6):

- If $T_\beta \neq \emptyset$ (see Figure 6(a)) then:
  - if $A < 1$, then place Γ_β one unit above Γ_α such that the left boundaries of Γ_β and Γ_α are aligned.
  - If $A \geq 1$, then first rotate Γ_β by 90° and then flip it vertically, then place Γ_β one unit to the left of Γ_α such that the top boundaries of Γ_β and Γ_α are aligned.

  Draw edge $(p(u), u)$.
- If $T_\beta = \emptyset$, then Γ_B is same as Γ_α (see Figure 6(b)).

Γ is constructed from Γ_A, Γ_B, and Γ_C as follows (see Figure 3):

- In Subcase (a), Γ is constructed as shown in Figure 3(a):
  - If $A < 1$, stack Γ_A, Γ_B, and Γ_C one above the other, such that they are separated by unit distance from each other, and the left boundaries of Γ_A and Γ_C are aligned with each other and are placed one unit to the left of the left boundary of Γ_B. Place a in the same vertical channel as o and $l(a)$, and in the same horizontal channel as $r(a)$.

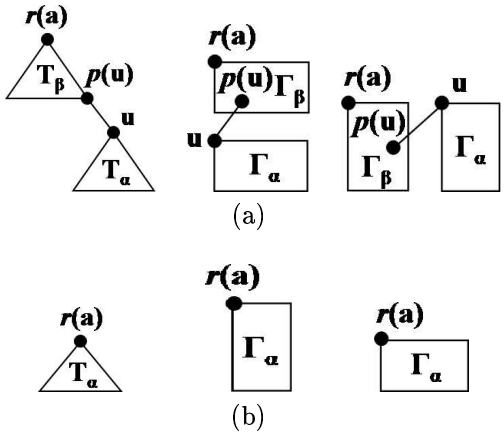


Figure 6: Drawing T_B when: (a) $T_\beta \neq \emptyset$, and (b) $T_\beta = \emptyset$. For each case, we first show the structure of T_B for that case, then its drawing when $A < 1$, and then its drawing when $A \geq 1$. In Case (a), for simplicity, $p(u)$ is shown to be in the interior of Γ_β, but actually, it is either same as $r(a)$, or if $A < 1$ ($A \geq 1$), then is placed on the bottom (right) boundary of Γ_β. For simplicity, we have shown Γ_β and Γ_α as identically sized boxes, but in actuality, their sizes may be different.

  - If $A \geq 1$, then first rotate Γ_A by 90°, and then flip it vertically. Then, place Γ_A, Γ_C, and Γ_B from left-to-right in that order, separated by unit distances, such that the top boundaries of Γ_A and Γ_B are aligned with each other, and are one unit above the top boundary of Γ_C. Then, move Γ_C down until u^* becomes the lowest node of Γ. Place a in the same vertical channel as $l(a)$ and in the same horizontal channel as o and $r(a)$.

  Draw edges $(p(a), a)$, $(a, r(a))$, and $(a, l(a))$.

- The drawing procedure for Subcase (b) is similar to the one in Subcase (a), except that we also flip Γ_C vertically (see Figure 3(b)).
- In Subcase (c), Γ is constructed as shown in Figure 3(c):
  - If $A < 1$, then first flip Γ_B vertically, and then flip it horizontally, so that its root $r(a)$ gets placed at its lower-right corner. Then, first rotate Γ_A by 90°, and then flip it vertically. Next, place Γ_A above Γ_B at unit distance, such that their left boundaries are aligned. Next move node $p(a)$ (which is the link node of T_A) to the right until it is either to the right of, or aligned with the right boundary of Γ_B (since Γ_A is a feasible drawing of T_A, by Property 2, as given in Section 4, moving $p(a)$ in this manner will not create any edge-crossings). Place u^* in the same horizontal channel as $r(a)$ and one unit to the right of $p(a)$.
  - If $A \geq 1$, then first rotate Γ_A by 90°, and then flip it vertically. Flip Γ_B vertically. Then, place Γ_A, u^*, and Γ_B left-to-right in that order separated by unit distances, such that the top boundaries of Γ_A and Γ_B are aligned, and u^* is placed in the same horizontal channel as the bottom boundary of the drawing among Γ_A and Γ_B with greater height.

  Draw edges $(p(a), a)$ and $(a, r(a))$ (i.e., the edges $(p(a), u^*)$ and $(u^*, r(a))$ because in this case, $u^* = a$).
- In Subcase (d), Γ is constructed as shown in Figure 3(d):
  - If $A < 1$, then first flip Γ_B vertically, then place Γ_A one unit above Γ_B, such that the left boundary of Γ_A is one unit to the left of the left boundary of Γ_B. Place u^* in the same vertical channel as o and in the same horizontal channel as $r(a)$.
  - If $A \geq 1$, then first flip Γ_B vertically, then place Γ_A one unit to the left of Γ_B, such that their top boundaries are aligned. Next, move Γ_B down until its bottom boundary is at least one unit below the bottom boundary of Γ_A. Place u^* in the same vertical channel as o and in the same horizontal channel as $r(a)$.

  Draw edges (o, u^*) and $(u^*, r(a))$ (i.e., the edges $(p(a), a)$ and $(a, r(a))$ because in this case, $o = p(a)$

and $u^* = a$). Note that, since Γ_A is a feasible drawing of T_A, from Property 3 (see Section 4), drawing (o, u^*) will not create any edge-crossings.

- In Subcase (e), for both $A < 1$ and $A \geq 1$, place node o one unit above and one unit left of Γ_B (see Figure 3(e)). Draw edge $(a, r(a))$ (i.e., the edge $(o, r(o))$ because in this case, $a = o$).
- The drawing procedure in Subcase (f) is similar to the one in Subcase (a), except that we do not have Γ_A here (see Figure 3(f)).
- The drawing procedure in Subcase (g) is similar to the one in Subcase (f), except that we also flip Γ_C vertically (see Figure 3(g)).

In Case 2, we construct Γ by composing Γ_A, Γ_B, and Γ_C, as follows (see Figure 4):

- The drawing procedures in Subcases (a) and (e) are similar to those in Subcases (a) and (f), respectively, of Case 1 (see Figures 4(a,e)).
- In Subcase (c), Γ is constructed as shown in Figure 4(c):
  - If $A > 1$, we place Γ_A one unit above Γ_C, such that the left boundary of Γ_C is one unit to the right of the left boundary of Γ_A. Place u in the same vertical channel as o and in the same horizontal channel as v.
  - If $A \geq 1$, then first rotate Γ_A by 90°, and then flip it vertically. Then, place Γ_A one unit to the left of Γ_C, such that the top boundary of Γ_C is one unit below the top boundary of Γ_A. Then, move Γ_C down until u^* becomes the lowest node of Γ. Place u in the same vertical channel as v and in the same horizontal channel as o.

  Draw edges $(p(u), u)$, and (u, v).
- The drawing procedure (see Figure 4(g)) in Subcase (g) is similar to that in Case (b) of drawing T_α (see Figure 5(b)).
- The drawing procedures in Subcases (b), (d), (f), and (h) are similar to those in Subcases (a), (c), (e), and (g), respectively (see Figures 4(b,d,f,h)), except that we also flip Γ_C vertically.

5.5 Proof of Correctness

Lemma 1 (Planarity) *Given a binary tree T with a link node u^*, Algorithm* DrawTree *will construct a feasible drawing Γ of T.*

Proof: We can easily prove using induction over the number of nodes n in T that Γ is a feasible drawing:
Base Case ($n = 1$): Γ consists of exactly one node and is trivially a feasible drawing.
Induction ($n > 1$): Consider Case 1. By the inductive hypothesis, the drawing constructed of each partial tree of T is a feasible drawing.

From Figure 5, it can be easily seen that in both the cases, $A < 1$ and $A \geq 1$, Γ_α is a planar drawing, and the root of T_α is placed at its top-left corner.

From Figure 6, it can be easily seen that in both the cases, $A < 1$ and $A \geq 1$, $r(a)$ is placed at the top-left corner of Γ_B. Note that because Γ_β is a feasible drawing of T_β and $p(u)$ is its link node, $p(u)$ is either at the bottom of Γ_β (from Property 2, see Section 4), or at the top-left corner of Γ_β and no other edge or node of T_β is placed on, or crosses the vertical channel occupied by it (Properties 1 and 3, see Section 4). Hence, in Figure 6(a), in the case $A < 1$, drawing edge $(p(u), u)$ will not cause any edge crossings. Also, in Figure 6(a), in the case $A \geq 1$, drawing edge $(p(u), u)$ will not cause any edge crossings because after rotating Γ_β by 90° and flipping it vertically, $p(u)$ will either be at the right boundary of Γ_β (because of Property 2), or at the top-left corner of Γ_β and no other edge or node of T_β will be placed on, or cross the horizontal channel occupied by it (because of Properties 1 and 3). It therefore follows that in both the cases, $A < 1$ and $A \geq 1$, Γ_B will be a planar drawing.

Finally, by considering each of the seven subcases shown in Figure 3 one-by-one, we can show that Γ is a feasible drawing of T:

- *Subcase (a):* See Figure 3(a). Γ_A is a feasible drawing of T_A, and $p(a)$ is the link node of T_A. Hence, $p(a)$ is either at the bottom of Γ_A (from Property 2), or is at the top-left corner of Γ_A, and no other edge or node of T_A is placed on, or crosses the horizontal and vertical channels occupied by it (from Properties 1 and 3). Hence, in the case $A < 1$, drawing edge $(p(a), a)$ will not create any edge-

crossings. In the case $A \geq 1$ also, drawing edge $(p(a), a)$ will not create any edge-crossings because after rotating Γ_A by 90° and flipping it vertically, $p(a)$ will either be at the right boundary of Γ_A (because of Property 2), or at the top-left corner of Γ_β and no other edge or node of T_A will be placed on, or cross the horizontal channel occupied by it (because of Properties 1 and 3).
Nodes $r(a)$ and $l(a)$ are placed at the top-left corner of Γ_B and Γ_C, respectively. Hence, drawing edges $(a, r(a))$ and $(a, l(a))$ will not create any edge-crossings in both the cases, $A < 1$ and $A \geq 1$.
In both the cases, $A < 1$ and $A \geq 1$, o gets placed at the top-left corner of Γ. Hence, Γ satisfies Property 1.
Since $u^* \neq o$, Property 3 is satisfied by Γ vacuously.
We now show that Property 2 is satisfied by Γ. In both the cases, $A < 1$ and $A \geq 1$, u^* gets placed at the bottom of Γ. Γ_C is a feasible drawing, $l(a)$ is the root of T_C, and $u^* \neq l(a)$. Hence, from Property 2, we can move u^* downwards in its vertical channel by any distance without causing any edge-crossings in Γ_C. Hence, in Γ also, we can move u^* downwards in its vertical channel by any distance without causing any edge-crossings in Γ. Thus, Property 2 is satisfied by Γ.
We therefore conclude that in both the cases, $A < 1$ and $A \geq 1$, Γ is a feasible drawing of T.

- *Subcase (b):* See Figure 3(b). The proof is similar to the one for Subcase (a), except that in this case, because $u^* = l(a)$, we use the fact that Γ_C satisfies Property 3 to prove that Γ satisfies Property 2. To elaborate, since $u^* = l(a)$, $l(a)$ is the root of Γ_C, and Γ_C is a feasible drawing, from Property 3, we can move u^* *upwards* in its vertical channel by any distance without causing any edge-crossings in Γ_C. We flip Γ_C vertically before placing it in Γ. Hence, it follows that in Γ, we can move u^* *downwards* in its vertical channel by any distance without causing any edge-crossings in Γ.
- *Subcase (c):* See Figure 3(c). Γ_A is a feasible drawing of T_A, $p(a)$ is the link node of T_A, and $p(a) \neq o$. Hence, from Property 2, $p(a)$ is located at the bottom of Γ_A. Rotating Γ_A by 90° and flipping it vertically will move $p(a)$ to the right boundary of Γ_A. Moving $p(a)$ to the right until it is either to the right of, or aligned with the right boundary of Γ_B will not cause any edge-crossings because of Property 2. It can be easily seen that in both the cases, $A < 1$ and $A \geq 1$, drawing edges $(p(a), u^*)$ and $(u^*, r(a))$ will not create any edge-crossings, and Γ will be a feasible drawing of T.
- *Subcase (d):* See Figure 3(d). Γ_A is a feasible drawing of T_A, $p(a)$ is the link node of T_A, and $p(a) = o$. Hence, from Properties 1 and 3, $p(a)$ is at the top-left corner of Γ_A, and no other edge or node of T_A is placed on, or crosses the horizontal and vertical channels occupied by it. Hence, in both the cases, $A < 1$ and $A \geq 1$, drawing edge $(p(a), u^*)$ will not create any edge-crossings, and Γ will be a feasible drawing of T.
- *Subcase (e):* See Figure 3(e). Because $r(a)$ is placed at the top-left corner of Γ_B, drawing edge $(a, r(a))$ will not cause any edge-crossings in both the cases, $A < 1$ and $A \geq 1$. It can be easily seen that Γ is a feasible drawing of T in both the cases when $A < 1$ and $A \geq 1$.
- *Subcase (f):* See Figure 3(f). It is straightforward to see that Γ is a feasible drawing of T in both the cases, $A < 1$ and $A \geq 1$.
- *Subcase (g):* See Figure 3(g). Γ_C is a feasible drawing of T_C, u^* is the link node of T_C, and u^* is also the root of T_C. Hence, from Properties 1 and 3, u^* is at the top-left corner of Γ_C, and no other edge or node of T_C is placed on, or crosses the horizontal and vertical channels occupied by it. Flipping Γ_C vertically will move u^* to the bottom-left corner of Γ_C and no other edge or node of T_C will be on or crosses the vertical channel occupied by it. Hence, drawing edge (o, u^*) will not create any edge-crossings. From Property 3, we can move u^* upwards in its vertical channel by any distance without causing any edge-crossings in Γ_C. We flip Γ_C vertically before placing it in Γ. Hence, in Γ, we can move u^* downwards in its vertical channel by any distance without causing any edge-crossings in Γ. It therefore follows that Γ is a feasible drawing of T.

Using a similar reasoning, we can show that in Case 2 also, Γ is a feasible drawing of T. □

Lemma 2 (Time) *Given an n-node binary tree T with a link node u^*, Algorithm* DrawTree *will construct a drawing Γ of T in $O(n \log n)$ time.*

Proof: From Theorem 1, each partial tree into which Algorithm *DrawTree* would split T will have at most $(2/3)n$ nodes in it. Hence, it follows that the depth of the recursion for Algorithm *DrawTree* is $O(\log n)$.

At the first recursive level, the algorithm will split T into partial trees, assign aspect ratios to the partial trees and compose the drawings of the partial trees to construct a drawing of T. At the next recursive level, it will split all of these partial trees into smaller partial trees, assign aspect ratios to these smaller partial trees, and compose the drawings of these smaller partial trees to construct the drawings of all the partial trees. This process will continue until the bottommost recursive level is reached. At each recursive level, the algorithm takes $O(m)$ time to split a tree with m nodes into partial trees, assign aspect ratios to the partial trees, and compose the drawings of partial trees to construct a drawing of the tree. At each recursive level, the total number of nodes in all the trees that the algorithm considers for drawing is at most n. Hence, at each recursive level, the algorithm totally spends $O(n)$ time. Hence, the running time of the algorithm is $O(n) \cdot O(\log n) = O(n \log n)$. □

Lemma 3 *Let R be a rectangle with area D and aspect ratio A. Let W and H be the width and height, respectively, of R. Then, $W = \sqrt{AD}$ and $H = \sqrt{D/A}$.*

Proof: By the definition of aspect ratio, $A = W/H$. $D = WH = W(W/A) = W^2/A$. Hence, $W = \sqrt{AD}$. $H = W/A = \sqrt{AD}/A = \sqrt{D/A}$. □

Lemma 4 (Area) *Let T be a binary tree with a link node u^*. Let n be the number of nodes in T. Let ϵ and A be two numbers such that $0 < \epsilon < 1$, and A is in the range $[n^{-\epsilon}, n^{\epsilon}]$. Given T, ϵ, and A as input, Algorithm* DrawTree *will construct a drawing Γ of T that can fit inside a rectangle R with $O(n)$ area and aspect ratio A.*

Proof: Let $D(n)$ be the area of R. We will prove, using induction over n, that $D(n) = O(n)$. More specifically, we will prove that $D(n) \leq c_1 n - c_2 n^{\beta}$ for all $n \geq n_0$, where n_0, c_1, c_2, β are some positive constants and $\beta < 1$.

We now give the proof for the case when $A \geq 1$ (the proof for the case $A < 1$ is symmetrical). Algorithm *DrawTree* will split T into at most 5 partial trees. Let T_k be a non-empty partial tree of T, where T_k is one of $T_A, T_\beta, T_1, T_2, T_C$ in Case 1, and is one of T_A, T_B, T_C in Case 2. Let n_k be the number of nodes in T_k, and let $x_k = n_k/n$. Let $P_k = c_1 n - c_2 n^{\beta}/x_k^{1-\beta}$. From Theorem 1, it follows that $n_k \leq (2/3)n$, and hence, $x_k \leq 2/3$. Hence, $P_k \leq c_1 n - c_2 n^{\beta}/(2/3)^{1-\beta} = c_1 n - c_2 n^{\beta}(3/2)^{1-\beta}$. Let $P' = c_1 n - c_2 n^{\beta}(3/2)^{1-\beta}$. Thus, $P_k \leq P'$.

From the inductive hypothesis, Algorithm *DrawTree* will construct a drawing Γ_k of T_k that can fit inside a rectangle R_k with aspect ratio A_k and area $D(n_k)$, where A_k is as defined in Section 5.2, and $D(n_k) \leq c_1 n_k - c_2 n_k^{\beta}$. Since $x_k = n_k/n$, $D(n_k) \leq c_1 n_k - c_2 n_k^{\beta} = c_1 x_k n - c_2 (x_k n)^{\beta} = x_k (c_1 n - c_2 n^{\beta}/x_k^{1-\beta}) = x_k P_k \leq x_k P'$.

Let W_k and H_k be the width and height, respectively, of R_k. We now compute the values of W_k and H_k in terms of A, P', x_k, n, and ϵ. We have two cases:

- *T_k is a small partial tree of T:* Then, $n_k < (n/A)^{1/(1+\epsilon)}$, and also, as explained in Section 5.2, $A_k = 1/n_k^{\epsilon}$. From Lemma 3, we have that $W_k = \sqrt{A_k D(n_k)} \leq \sqrt{(1/n_k^{\epsilon})(x_k P')} = \sqrt{(1/n_k^{\epsilon})(n_k/n)P'} = \sqrt{n_k^{1-\epsilon} P'/n}$. Since $n_k < (n/A)^{1/(1+\epsilon)}$, it follows that $W_k < \sqrt{(n/A)^{(1-\epsilon)/(1+\epsilon)} P'/n} = \sqrt{(1/A^{(1-\epsilon)/(1+\epsilon)}) P'/n^{2\epsilon/(1+\epsilon)}} \leq \sqrt{P'/n^{2\epsilon/(1+\epsilon)}}$, since $A \geq 1$.

  From Lemma 3, $H_k = \sqrt{D(n_k)/A_k} \leq \sqrt{x_k P'/(1/n_k^{\epsilon})} = \sqrt{(n_k/n)P' n_k^{\epsilon}} = \sqrt{n_k^{1+\epsilon} P'/n}$. Since $n_k < (n/A)^{1/(1+\epsilon)}$, $H_k < \sqrt{(n/A)^{(1+\epsilon)/(1+\epsilon)} P'/n} = \sqrt{(n/A)P'/n} = \sqrt{P'/A}$.
- *T_k is a large partial tree of T:* Then, as explained in Section 5.2, $A_k = x_k A$. From Lemma 3, $W_k = \sqrt{A_k D(n_k)} \leq \sqrt{x_k A x_k P'} = x_k \sqrt{AP'}$.
  From Lemma 3, $H_k = \sqrt{D(n_k)/A_k} \leq \sqrt{x_k P'/(x_k A)} = \sqrt{P'/A}$.

In Step *Compose Drawings*, we use at most two additional horizontal channels and at most one additional vertical channel while combining the drawings of the partial trees to construct a drawing Γ of T. For example, in Case 1(e), if $u \neq a$ and $T_1 = \emptyset$, then we use one additional horizontal channel and one additional vertical channel for placing a (see Figure 3(e)), and one additional horizontal channel for placing u (see Figure 5(b)).

Hence, Γ can fit inside a rectangle R' with width W' and height H', respectively, where,

$$H' \leq \max_{T_k \text{ is a partial tree of } T} \{H_k\} + 2 \leq \sqrt{P'/A} + 2,$$

and

$$\begin{aligned} W' &\leq \sum_{T_k \text{ is a large partial tree}} W_k + \sum_{T_k \text{ is a small partial tree}} W_k + 1 \\ &\leq \sum_{T_k \text{ is a large partial tree}} x_k\sqrt{AP'} + \sum_{T_k \text{ is a small partial tree}} \sqrt{P'/n^{2\epsilon/(1+\epsilon)}} + 1 \\ &\leq \sqrt{AP'} + 5\sqrt{P'/n^{2\epsilon/(1+\epsilon)}} + 1 \end{aligned}$$

(because $\sum_{T_k \text{ is a large partial tree}} x_k \leq 1$, and T has at most 5 partial trees).

R' might not have aspect ratio equal to A, but it is contained within a rectangle R with aspect ratio A, area $D(n)$, width W, and height H, where

$$W = \sqrt{AP'} + 5\sqrt{P'/n^{2\epsilon/(1+\epsilon)}} + 1 + 2A,$$

and

$$H = \sqrt{P'/A} + 2 + (5/A)\sqrt{P'/n^{2\epsilon/(1+\epsilon)}} + 1/A$$

Hence, $D(n) = WH = (\sqrt{AP'} + 5\sqrt{P'/n^{2\epsilon/(1+\epsilon)}} + 1 + 2A)(\sqrt{P'/A} + 2 + (5/A)\sqrt{P'/n^{2\epsilon/(1+\epsilon)}} + 1/A) \leq P' + c_3P'/\sqrt{An^{2\epsilon/(1+\epsilon)}} + c_4\sqrt{AP'} + c_5P'/(An^{2\epsilon/(1+\epsilon)}) + c_6\sqrt{P'/n^{2\epsilon/(1+\epsilon)}} + c_7A + c_8 + c_9/A + c_{10}\sqrt{P'/A} + c_{11}\sqrt{P'/n^{2\epsilon/(1+\epsilon)}}/A$, where $c_3, c_4, \ldots, c_{11}$ are some constants.

Since, $1 \leq A \leq n^\epsilon$, we have that

$$\begin{aligned} D(n) \leq\ & P' + c_3P'/\sqrt{n^{2\epsilon/(1+\epsilon)}} + c_4\sqrt{n^\epsilon P'} + c_5P'/n^{2\epsilon/(1+\epsilon)} + c_6\sqrt{P'/n^{2\epsilon/(1+\epsilon)}} \\ & + c_7n^\epsilon + c_8 + c_9 + c_{10}\sqrt{P'} + c_{11}\sqrt{P'/n^{2\epsilon/(1+\epsilon)}} \end{aligned}$$

Since $P' < c_1n$,

$$\begin{aligned} D(n) &< P' + c_3c_1n/\sqrt{n^{2\epsilon/(1+\epsilon)}} + c_4\sqrt{n^\epsilon c_1n} + c_5c_1n/n^{2\epsilon/(1+\epsilon)} \\ &\quad + c_6\sqrt{c_1n/n^{2\epsilon/(1+\epsilon)}} + c_7n^\epsilon + c_8 + c_9 + c_{10}\sqrt{c_1}n^{1/2} \\ &\quad + c_{11}\sqrt{c_1n/n^{2\epsilon/(1+\epsilon)}} \\ &\leq P' + c_3c_1n^{1/(1+\epsilon)} + c_4\sqrt{c_1}n^{(1+\epsilon)/2} + c_5c_1n^{(1-\epsilon)/(1+\epsilon)} \\ &\quad + c_6\sqrt{c_1}n^{(1-\epsilon)/(2(1+\epsilon))} + c_7n^\epsilon + c_8 + c_9 + c_{10}\sqrt{c_1}n^{1/2} \\ &\quad + c_{11}\sqrt{c_1}n^{(1-\epsilon)/(2(1+\epsilon))} \\ &\leq P' + c_{12}n^{1/(1+\epsilon)} + c_{13}n^{(1+\epsilon)/2} \end{aligned}$$

where c_{12} and c_{13} are some constants (because, since $0 < \epsilon < 1$, $(1-\epsilon)/(2(1+\epsilon)) < (1-\epsilon)/(1+\epsilon) < 1/(1+\epsilon)$, $\epsilon < (1+\epsilon)/2$, and $1/2 < (1+\epsilon)/2$).

$P' = c_1n - c_2n^\beta(3/2)^{1-\beta} = c_1n - c_2n^\beta(1+c_{14})$, where c_{14} is a constant such that $1 + c_{14} = (3/2)^{1-\beta}$.

Hence, $D(n) \leq c_1n - c_2n^\beta(1+c_{14}) + c_{12}n^{1/(1+\epsilon)} + c_{13}n^{(1+\epsilon)/2} = c_1n - c_2n^\beta - (c_2c_{14}n^\beta - c_{12}n^{1/(1+\epsilon)} - c_{13}n^{(1+\epsilon)/2})$. Thus, for a large enough constant n_0, and large enough β, where $1 > \beta > \max\{1/(1+\epsilon), (1+\epsilon)/2\}$, for all $n \geq n_0$, $c_2c_{14}n^\beta - c_{12}n^{1/(1+\epsilon)} - c_{13}n^{(1+\epsilon)/2} \geq 0$, and hence $D(n) \leq c_1n - c_2n^\beta$.

The proof for the case $A < 1$ uses the same reasoning as for the case $A \geq 1$. With T_k, R_k, W_k, H_k, R', W', H', R, W, and H defined as above, and A_k as defined in Section 5.2, we get the following values for W_k, H_k, W', H', W, H, and $D(n)$:

$$\begin{aligned} W_k &\leq \sqrt{AP'} \\ H_k &\leq \sqrt{P'/n^{2\epsilon/(1+\epsilon)}} \text{ if } T_k \text{ is a small partial tree} \end{aligned}$$

$$
\begin{aligned}
& \leq x_k\sqrt{P'/A} \quad \text{if } T_k \text{ is a large partial tree} \\
W' &\leq \sqrt{AP'} + 2 \\
H' &\leq \sqrt{P'/A} + 5\sqrt{P'/n^{2\epsilon/(1+\epsilon)}} + 1 \\
W &\leq \sqrt{AP'} + 2 + 5A\sqrt{P'/n^{2\epsilon/(1+\epsilon)}} + A \\
H &\leq \sqrt{P'/A} + 5\sqrt{P'/n^{2\epsilon/(1+\epsilon)}} + 1 + 2/A \\
D(n) &\leq P' + c_{12}n^{1/(1+\epsilon)} + c_{13}n^{(1+\epsilon)/2}
\end{aligned}
$$

where c_{12} and c_{13} are the same constants as in the case $A \geq 1$. Therefore, $D(n) \leq c_1 n - c_2 n^{\beta}$ for $A < 1$ too. (Notice that in the values that we get above for W_k, H_k, W', H', W, and H, if we replace A by $1/A$, exchange W_k with H_k, exchange W' with H', and exchange W with H, we will get the same values for W_k, H_k, W', H', W, and H as in the case $A \geq 1$. This basically reflects the fact that the cases $A > 1$ and $A < 1$ are symmetrical to each other.) □

Theorem 2 (Main Theorem) *Let T be a binary tree with n nodes. Given two numbers A and ϵ, where ϵ is a constant, such that $0 < \epsilon < 1$, and $n^{-\epsilon} \leq A \leq n^{\epsilon}$, we can construct in $O(n \log n)$ time, a planar straight-line grid drawing of T with $O(n)$ area and aspect ratio A.*

Proof: Arbitrarily select a node of T that has at most one child (for example, we can select any leaf), and designate it the link node of T. Construct a drawing Γ of T by invoking Algorithm *DrawTree* with T, A, and ϵ as input. From Lemmas 1, 2, and 4, Γ will be a planar straight-line grid drawing contained entirely within a rectangle with $O(n)$ area and aspect ratio A. □

Corollary 1 *Let T be a binary tree with n nodes. We can construct in $O(n \log n)$ time, a planar straight-line grid drawing of T with optimal (equal to $O(n)$) area and optimal aspect ratio (equal to 1).*

Proof: Immediate from Theorem 2, with $A = 1$, and ϵ any constant, such that $0 < \epsilon < 1$. □

6 Experimental Results

We have implemented the algorithm using C++. The implementation consists of about 2100 lines of code. We have also experimentally evaluated the algorithm on two types of binary trees, namely, randomly-generated, consisting of up to 50,000 nodes, and complete, consisting of up to $65,535 = 2^{16} - 1$ nodes.

Each randomly-generated binary tree T_n with n nodes was generated by generating a sequence $T_0, T_1, \ldots, T_n$ of binary trees, where T_0 is the empty tree, and T_i was generated from T_{i-1} by inserting a new leaf v_i into it. The position, where v_i is inserted in T_{i-1}, is determined by traversing a path $p = u_0 u_1 \ldots u_m$ of T_{i-1}, where u_0 is the root of T_{i-1}, and u_m has at most one child. More precisely, we start at the root u_0, and in the general step, assuming that we have already traversed the subpath $u_0 u_1 \ldots u_{i-1}$, we flip a coin. If "head" comes up, then if u_{i-1} has a left child c, then we set $u_i = c$, and move to u_i, otherwise we make v_i the left child of u_{i-1}, and stop. If "tail" comes up, then if u_{i-1} has a right child c, then we set $u_i = c$, and move to u_i, otherwise we make v_i the right child of u_{i-1}, and stop.

Recall that the algorithm takes three values as input: a binary tree T with n nodes, a number ϵ, where $0 < \epsilon < 1$, and a number A in the range $[n^{-\epsilon}, n^{\epsilon}]$.

The performance criteria we have used to evaluate the algorithm is the ratio c of the area of the drawing constructed of a tree T, and the number of nodes in T. Recall that the area and aspect ratio of a drawing is defined as the area and aspect ratio, respectively, of its enclosing rectangle.

To evaluate the algorithm, we varied n up to $50,000$ for randomly-generated trees, and up to $65,535 = 2^{16} - 1$ for complete trees. For each n, we used five different values for ϵ, namely, 0.1, 0.25, 0.5, 0.75, and 0.9. For each (n, ϵ) pair, we used 20 different values of A uniformly distributed in the range $[1, n^{\epsilon}]$. The performance of the algorithm is symmetrical for $A < 1$ and $A > 1$. Hence, we varied A only from 1 through n^{ϵ}, not from $n^{-\epsilon}$ through n^{ϵ} (the only difference between $A < 1$ and $A > 1$ is that for $A < 1$ the algorithm constructs drawings with longer height than width, whereas for $A > 1$, it constructs drawings with longer width than height). Hence, in the rest of the section, we will assume that $A \geq 1$. For each type of tree (randomly-generated and complete), and for each triplet (n, A, ϵ), we generated three trees of that type. We

constructed a drawing of each tree using the algorithm, and computed the value of c. Next, we averaged the values of c obtained for the three trees to get a single value for each triplet (n, A, ϵ) for each tree-type.

Our experiments show that the value of c is generally small, and is at most 10 for randomly-generated, and at most 8 for complete trees. Figure 7, and Figure 8 show how c varies with n, A, and ϵ for randomly-generated, and complete trees, respectively.

We also discovered that c increases with A for a given n and ϵ. However, the rate of increase is very small. Consequently, for a given n and ϵ, the range for c over all the values of A is small (see Figure 7(b,d,f,h,j), and Figure 8(b,d,f,h,j)). For example, for $n = 10,000$, and $\epsilon = 0.5$, for randomly-generated trees, the range for c is $[4.2, 5.2]$.

Similarly, for a given n and A, c increases with ϵ.

Finally, we would like to comment that the aspect ratio of the drawing constructed is, in general, different from the input aspect ratio A. We computed the ratio r of the aspect ratio of the drawing constructed by the algorithm and input aspect ratio A. We discovered that r is close to 1 for $A = 1$, generally decreases as we increase A, and can get as low as 0.1 for $A = n^{\epsilon}$. However, we also discovered that for a large range of values for A, namely, $[1, \min\{n^{\epsilon}, n/\log^2 n\}]$, r stays within the range $[0.8, 1.5]$, and so is close to 1. Hence, even in applications, that require the drawing to be of exactly the same aspect ratio as A, we can obtain a drawing with small area and aspect ratio exactly equal to A by adding enough "white space" to the drawing constructed by our drawing algorithm. Adding the white space will increase the area of the drawing by a factor of at most $\max\{1/0.8, 1.5\} = 1.5$ (assuming that A is in the above-mentioned range). Hence, the area of the drawing will still be small.

References

[1] T. Chan, M. Goodrich, S. Rao Kosaraju, and R. Tamassia. Optimizing area and aspect ratio in straight-line orthogonal tree drawings. *Computational Geometry: Theory and Applications*, 23:153–162, 2002.

[2] P. Crescenzi, G. Di Battista, and A. Piperno. A note on optimal area algorithms for upward drawings of binary trees. *Comput. Geom. Theory Appl.*, 2:187–200, 1992.

[3] P. Crescenzi, P. Penna, and A. Piperno. Linear-area upward drawings of AVL trees. *Comput. Geom. Theory Appl.*, 9:25–42, 1998. (special issue on Graph Drawing, edited by G. Di Battista and R. Tamassia).

[4] G. Di Battista, P. Eades, R. Tamassia, and I. G. Tollis. *Graph Drawing.* Prentice Hall, Upper Saddle River, NJ, 1999.

[5] A. Garg, M. T. Goodrich, and R. Tamassia. Planar upward tree drawings with optimal area. *Internat. J. Comput. Geom. Appl.*, 6:333–356, 1996.

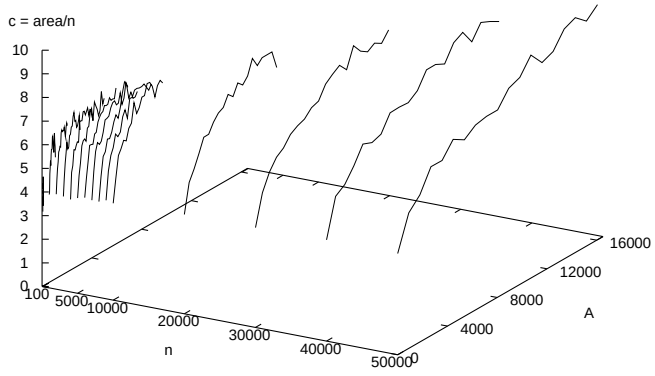
[6] A. Garg and A. Rusu. Straight-line drawings of binary trees with linear area and arbitrary aspect ratio. In *Proc. 10th International Symposium on Graph Drawing (GD 2002)*, volume 2528 of *Lecture Notes in Computer Science*, pages 320–331. Springer-Verlag, 2002.

[7] C. E. Leiserson. Area-efficient graph layouts (for VLSI). In *Proc. 21st Annu. IEEE Sympos. Found. Comput. Sci.*, pages 270–281, 1980.

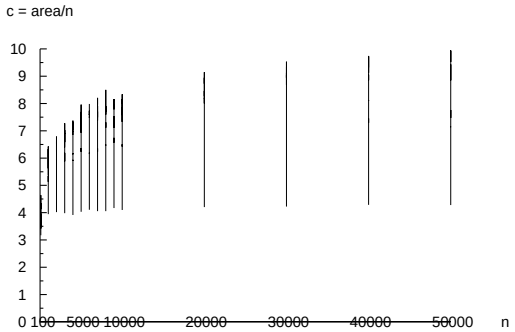
[8] C.-S. Shin, S.K. Kim, S.-H. Kim, and K.-Y. Chwa. Area-efficient algorithms for straight-line tree drawings. *Comput. Geom. Theory Appl.*, 15:175–202, 2000.

[9] L. Trevisan. A note on minimum-area upward drawing of complete and Fibonacci trees. *Inform. Process. Lett.*, 57(5):231–236, 1996.

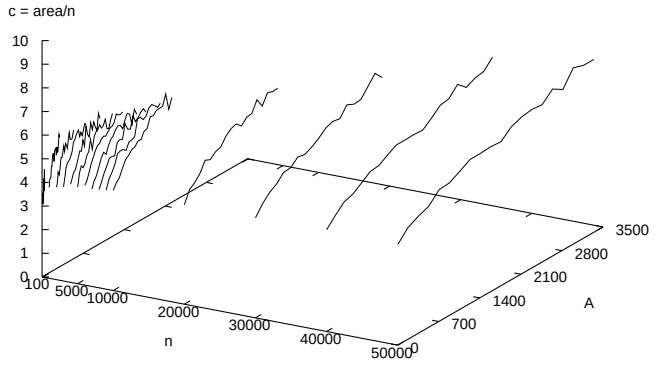
[10] L. Valiant. Universality considerations in VLSI circuits. *IEEE Trans. Comput.*, C-30(2):135–140, 1981.



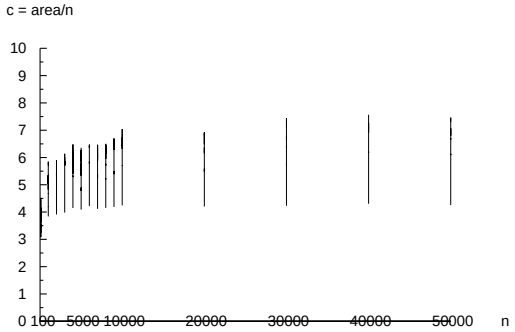
(a) $\epsilon = 0.9$



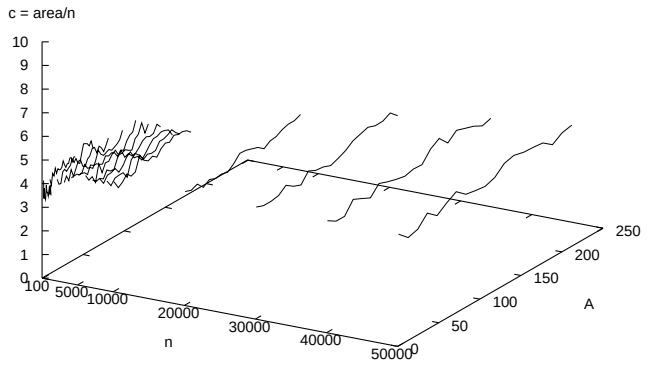
(b) $\epsilon = 0.9$



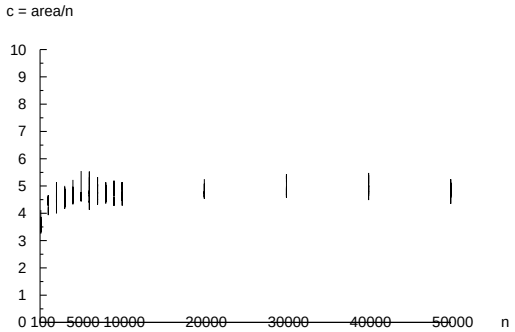
(c) $\epsilon = 0.75$



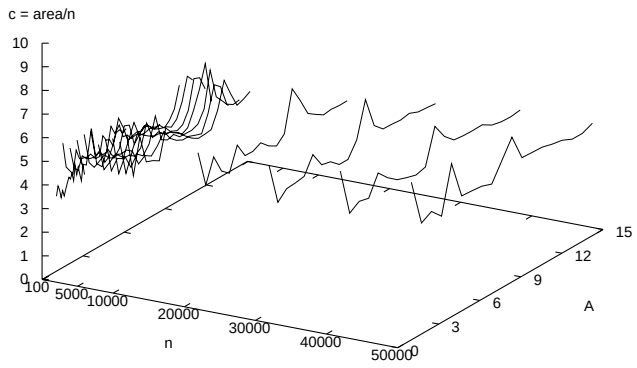
(d) $\epsilon = 0.75$



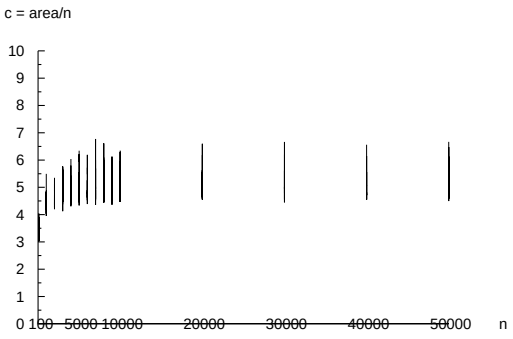
(e) $\epsilon = 0.5$



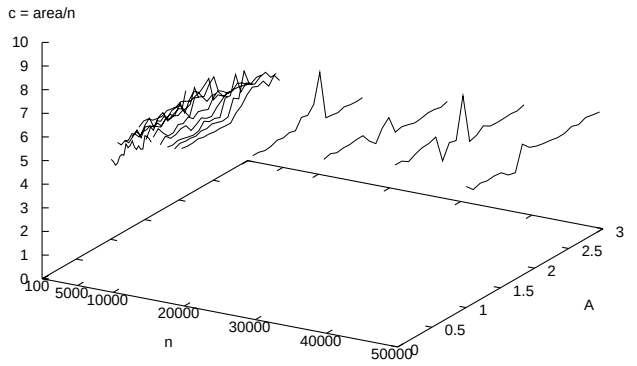
(f) $\epsilon = 0.5$



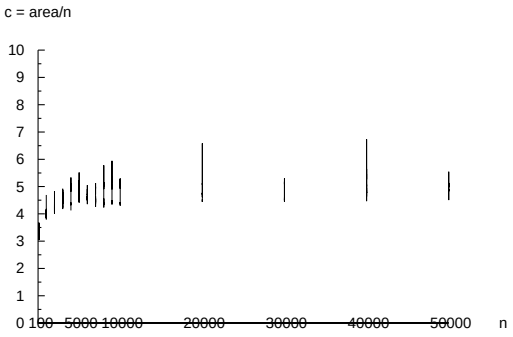
(g) $\epsilon = 0.25$



(h) $\epsilon = 0.25$

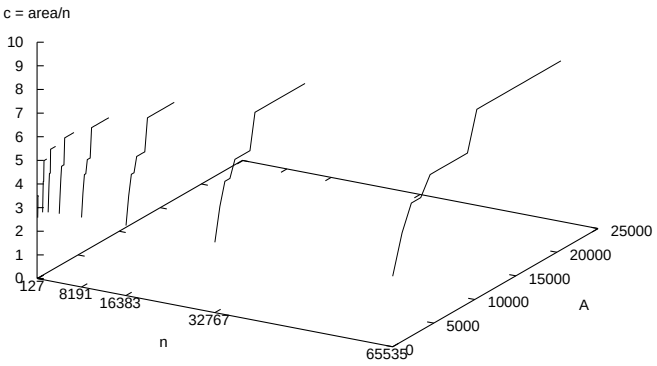


(i) $\epsilon = 0.1$

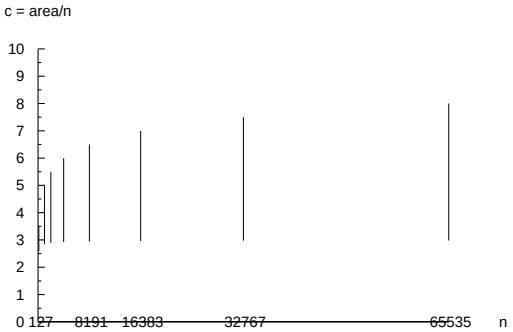


(j) $\epsilon = 0.1$

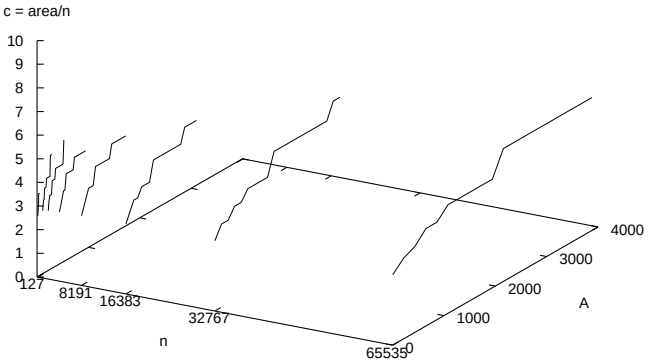
Figure 7: Performance of the algorithm, as given by the value of c, for drawing a randomly-generated binary tree T with different values of A and ϵ, where *c=area of drawing/number of nodes n in T*: (a) $\epsilon = 0.9$, (c) $\epsilon = 0.75$, (e) $\epsilon = 0.5$, (g) $\epsilon = 0.25$, and (i) $\epsilon = 0.1$. Figures (b), (d), (f), (h), and (j) contain the projections on the XZ-plane of the plots shown in Figures (a), (c), (e), (g), and (i), respectively, and show for each ϵ, the ranges for the values of c for different values of A for each n.



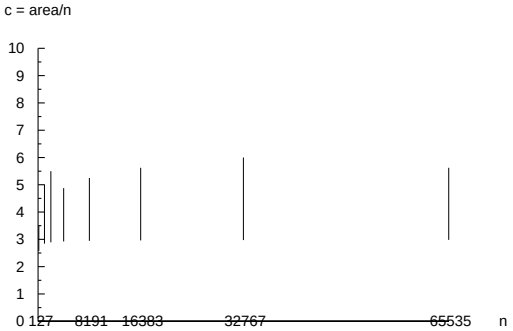
(a) $\epsilon = 0.9$



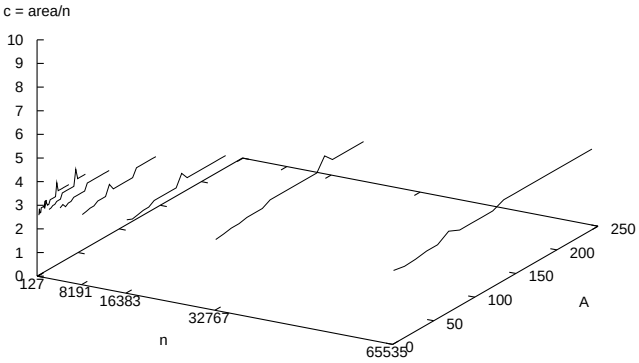
(b) $\epsilon = 0.9$



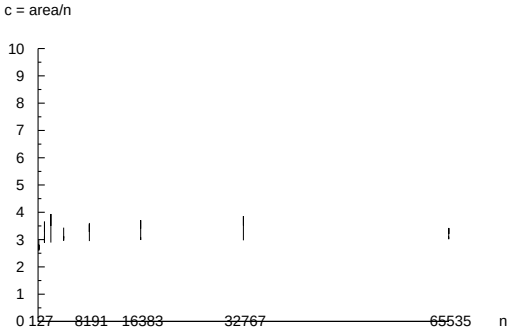
(c) $\epsilon = 0.75$



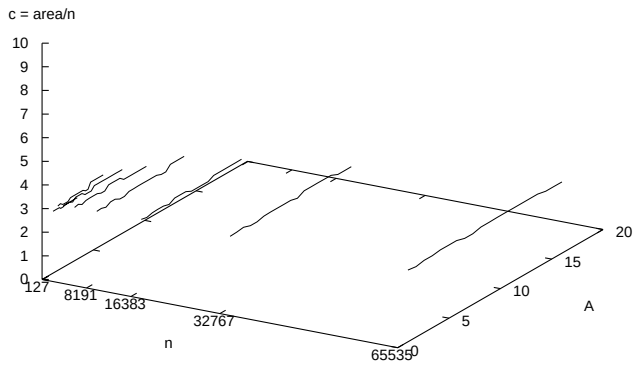
(d) $\epsilon = 0.75$



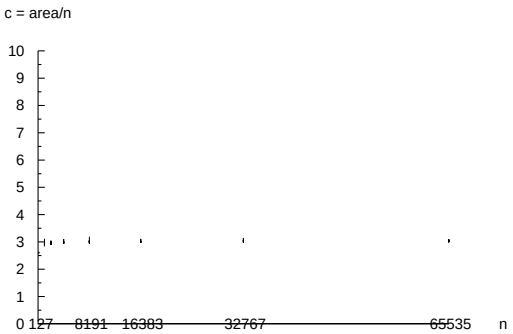
(e) $\epsilon = 0.5$



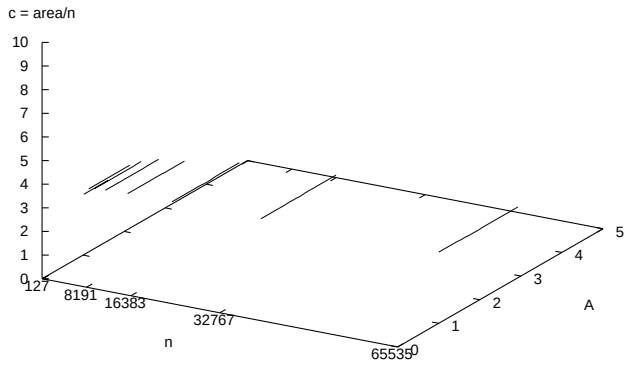
(f) $\epsilon = 0.5$



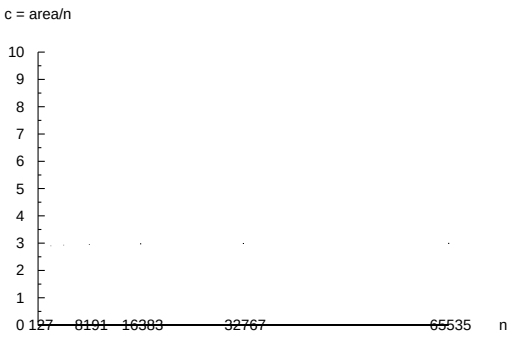
(g) $\epsilon = 0.25$



(h) $\epsilon = 0.25$



(i) $\epsilon = 0.1$



(j) $\epsilon = 0.1$

Figure 8: Performance of the algorithm, as given by the value of c, for drawing a complete binary tree T with different values of A and ϵ, where *c=area of drawing/number of nodes n in T*: (a) $\epsilon = 0.9$, (c) $\epsilon = 0.75$, (e) $\epsilon = 0.5$, (g) $\epsilon = 0.25$, and (i) $\epsilon = 0.1$. Figures (b), (d), (f), (h), and (j) contain the projections on the XZ-plane of the plots shown in Figures (a), (c), (e), (g), and (i), respectively, and show for each ϵ, the ranges for the values of c for different values of A for each n.