Improved Visibility Representation of Plane Graphs*

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Abstract

In a visibility representation (VR for short) of a plane graph G, each vertex of G is represented by a horizontal line segment such that the line segments representing any two adjacent vertices of G are joined by a vertical line segment. Rosenstiehl and Tarjan [6], Tamassia and Tollis [9] independently gave linear time VR algorithms for 2-connected plane graph. Using this approach, the height of the VR is bounded by $(n-1)$, the width is bounded by $(2n-5)$. After that, some work have been done to find a more compact VR. Kant and He [3] proved that a 4-connected plane graph has a VR with width bounded by $(n-1)$. Kant [2] reduced the width bound to $\lfloor \frac{3n-6}{2} \rfloor$ for general plane graphs. Recently, using a sophisticated greedy algorithm, Lin et. al. reduced the width bound to $\lfloor \frac{22n-42}{15} \rfloor$ [5].

In this paper, we prove that any plane graph G has a VR with width at most $\lfloor \frac{13n-24}{9} \rfloor$, which can be constructed by using the simple standard VR algorithm in [6, 9].

Keywords: Graph; Visibility representation

1 Introduction

A *visibility representation* (VR for short) of a plane graph G is a drawing of G, where the vertices of G are represented by non-overlapping horizontal segments (called *vertex segments*), and each edge of G is represented by a vertical line segment touching the vertex segments of its end vertices. The problem of computing a compact VR is important not only in algorithmic graph theory, but also in practical applications such as VLSI layout [6, 9].

Without loss of generality, we assume that G is a 2-connected plane graph with n vertices. It was shown in [6, 9] that a VR of a plane graph G can be obtained by a simple linear time algorithm from an *st-orientation* of G and the corresponding st-orientation of its dual graph G^*. Using this approach, the height of the VR is bounded by $(n-1)$ and the width of the VR is bounded by $(2n-5)$ [6, 9].

One of the main concerns for VR is the size of the representation. Some work have been done to reduce the width of the VR by carefully choosing a special st-orientation of G. Kant and He proved that every 4-connected plane graph G has a VR with width bounded by $(n-1)$ [3]. Based on this, Kant proved that every plane graph has a VR with width at most $\lfloor \frac{3n-6}{2} \rfloor$ [2]. Very recently, using a more sophisticated greedy algorithm, Lin et. al. reduced the width bound to $\lfloor \frac{22n-42}{15} \rfloor$ by choosing the best st-orientation from three st-orientations derived from a *Schnyder's realizer* of G [5]. Their algorithm runs in linear time and uses a sophisticated greedy approach.

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In this paper, we prove that every plane graph G has a VR with width at most $\lfloor \frac{13n-24}{9} \rfloor$ by applying the simple standard VR algorithm in [6, 9]. In order to obtain the new width bound, we first prove that, for a plane graph G with an st-orientation $\mathcal{O}$, the width of the VR constructed by the algorithm in [6, 9] is bounded by $|E| - Score_{\mathcal{O}}(G)$, where $Score_{\mathcal{O}}(G)$ is the sum of the minimum of the number of bigger neighbors and smaller neighbors of vertices in G. Then we prove that, among the three st-orientations from a Schnyder's realizer of a plane triangulation G, at least one st-orientation $\mathcal{O}$ has $Score_{\mathcal{O}}(G) \geq \lceil \frac{14n-30}{9} \rceil$. Thus we are able to obtain the new width bound $\lfloor \frac{13n-24}{9} \rfloor$.

The present paper is organized as follows. Section 2 introduces definitions and preliminary results. Section 3 proves the relation between the width of VR and the score of an st-orientation for a 2-connected plane graph. Section 4 presents the construction of a VR with width at most $\lfloor \frac{13n-24}{9} \rfloor$.

2 Preliminaries

In this section, we give definitions and preliminary results. $G = (V, E)$ denotes a graph with $n = |V|$ vertices and $m = |E|$ edges. The *degree* of a vertex $v \in V$, denoted by $deg_G(v)$, is the number of edges incident to v. If G is clearly understood, we simply write $deg(v)$ for $deg_G(v)$. A *planar graph* G is a graph which can be embedded on the plane without edge crossings. A *plane graph* is a planar graph with a fixed embedding. The embedding of a plane graph divides the plane into a number of regions, called *faces*. The unbounded region is the *exterior face*. Other regions are *interior faces*. The vertices and the edges on the exterior face are called *exterior vertices* and *exterior edges*. Other vertices and edges are called *interior vertices* and *interior edges*. A *path* P of G is a sequence of distinct vertices $u_1, u_2, \ldots, u_k$ such that $(u_i, u_{i+1}) \in E$ for $1 \leq i < k$. We also use P to denote the set of the edges in it. Each u_i for $1 < i < k$ is called *an internal vertex* of P. Furthermore, if $(u_k, u_1) \in E$, then $u_1, u_2, \ldots, u_k$ is called a *cycle*. We normally use C to denote a cycle and the set of the edges of it. If C has 3 vertices, it is called a *triangle*. A cycle C of G divides the plane into its interior region and exterior region. If all facial cycles of G are triangles, G is called a *plane triangulation*. We abbreviate the words "counterclockwise" and "clockwise" as ccw and cw respectively.

The dual graph $G^* = (V^*, E^*)$ of a plane graph G is defined as follows: For each face F of G, G^* has a node v_F. For each edge e in G, G^* has an edge $e^* = (v_{F_1}, v_{F_2})$ where F_1 and F_2 are the two faces of G with e on their common boundaries. e^* is called the *dual edge* of e. For each vertex $v \in V$, the dual face of v in G^* is denoted by v^*.

An *orientation* of a graph G is a digraph obtained from G by assigning a direction to each edge of G. We will use G to denote both the resulting digraph and the underlying undirected graph unless otherwise specified. (Its meaning will be clear from the context).

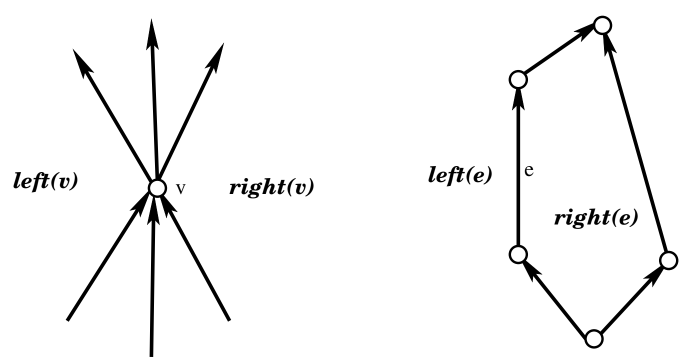

Figure 1: Properties of an st-orientation.

Let G be a plane graph with a *distinguished* exterior edge (s,t). An orientation of G is called an *st-orientation*, if it is acyclic with s as the only source and t as the only sink.

Let G be a plane graph and (s,t) a distinguished exterior edge of G. An *st-numbering* of G is a one-to-one mapping $\xi : V \rightarrow \{1, 2, \cdots, n\}$, such that $\xi(s) = 1$, $\xi(t) = n$, and each vertex $v \neq s, t$ has at least two neighbors u, w with $\xi(u) < \xi(v) < \xi(w)$, where u (w, resp.) is called a *smaller (bigger, resp.) neighbor* of v. Given an st-numbering ξ of G, we can orient G by directing each edge in E from its lower numbered end vertex to its higher numbered end vertex. The resulting orientation will be called the *orientation derived from* ξ. Obviously, this orientation is an st-orientation of G. On the other hand, if $G = (V, E)$ has an st-orientation $\mathcal{O}$, we can define a 1-1 mapping $\xi : V \rightarrow \{1, \cdots, n\}$ by using topological sort. It is easy to see that ξ is an st-numbering (where $\xi(s) = 1$ and $\xi(t) = n$) and the orientation derived from ξ is $\mathcal{O}$. From now on, we will interchangeably use the term an st-numbering of G and the term an st-orientation of G, where each edge of G is directed accordingly.

Given any st-orientation of G, for each vertex v, the incoming edges of v appear consecutively around v, and so do the outgoing edges of v. The face of G that separates the incoming edges of v from the outgoing edges of v in clockwise direction is denoted by $left(v)$. The face of G that separates the incoming edges of v from the outgoing edges of v is denoted by $right(v)$. The boundary of every face of G consists of two directed paths. For each edge e of G, the face on the left (right, resp.) side of e is denoted by $left(e)$ ($right(e)$, resp.) (See Fig. 1.)

Lempel et. al. [4] showed that for every 2-connected plane graph G and any exterior edge (s,t), there exists an st-numbering ξ of G. Thus, G has an st-orientation derived from ξ, with s as the only source and t as the only sink.

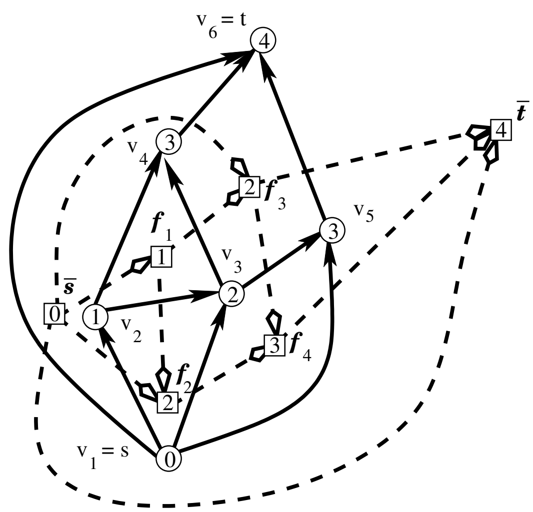

Figure 2: A 2-connected plane graph and one of its st-orientation $\mathcal{O}$.

Let G be a 2-connected plane graph and (s,t) be an exterior edge. Let $\mathcal{O}$ be an st-orientation of G, where s is the source and t is the sink. Consider the dual graph G^* of G. For each $e \in G$, we direct its dual edge e^* from the face on the left of e to the face on the right of e when we walk on e along its direction in $\mathcal{O}$. We then reverse the direction of $(s,t)^*$. It was shown in [6, 9] that this orientation is an st-orientation of G^* with $(s,t)^*$ as the distinguished exterior edge. We denote the source by $\bar{s}$, and the sink by $\bar{t}$. When we embed G and G^* on plane simultaneously, we fix t^* to be the exterior face of G^*. We will denote this orientation of G^* by $\mathcal{O}^*$ and call it the corresponding st-orientation of $\mathcal{O}$. For each vertex v of G, we define $dist(v)$ to be the length of the longest path from source s to v. For each node v^* in G^*, define $dist^*(v^*)$ to be the length of the longest directed path from the source $\bar{s}$ to v^*.

For example, in Fig. 2, the graph G drawn in solid lines is 2-connected with one st-orientation $\mathcal{O}$, G^* is drawn in dashed lines with $\mathcal{O}^*$. The vertices of G are represented by circles. The nodes of G^* are represented by squares. The numbers inside the circles (squares, resp.) are the $dist(v)$ of vertices in G ($dist^*(v^*)$ of nodes in G^*, resp.)

The following VR algorithm was given in [6, 9]:

Algorithm 1: Visibility Representation
Input: A 2-connected plane graph G.
Output: A VR of G.

1. Compute an st-orientation $\mathcal{O}$ of G.
2. Construct its dual G^* and the corresponding st-orientation $\mathcal{O}^*$ of G^*.
3. Compute $dist(v)$ for the vertices of G and $dist^*(v^*)$ for the nodes of G^*.
4. For each vertex v of G, do:
   If $v \neq s, t$, draw horizontal line between $(dist^*(left(v)), dist(v))$ and $(dist^*(right(v)) - 1, dist(v))$.
   If $v = s$ or t, draw horizontal line between $(0, d(v))$ and $(dist^*(\bar{t}), d(v))$.
5. For each edge (u, v) of G, draw vertical line between $(dist^*(left(u, v)), dist(u))$ and $(dist^*(left(u, v)), dist(v))$.

The correctness of the above VR drawing algorithm was proved in [6, 9]:

Lemma 1 *Let G be a 2-connected plane graph with n vertices. Let $\mathcal{O}$ be an st-orientation of G. Algorithm 1 produces a VR of G in linear time. The width of the VR is the length of the longest directed path in the corresponding st-orientation $\mathcal{O}^*$ of G^*.*

3 Bounding the Width of VR of a Plane Graph

We assume that G is a 2-connected plane graph in this section. In order to obtain a VR of G with reduced width, we need to find an st-orientation $\mathcal{O}$ of G such that the length of longest path in $\mathcal{O}^*$ of G^* is reduced. First, we introduce several concepts.

Definition 1 *Let G be a 2-connected plane graph, $\mathcal{O}$ be an st-orientation of G. Let G^* be the dual graph of G, $\mathcal{O}^*$ be the corresponding st-orientation of $\mathcal{O}$.*

1. *For any vertex $v \neq s, t$ of G, define:*
   $Hand(v) = \{(v, u) \in E \mid u \text{ is a bigger neighbor of } v\}$;
   $Foot(v) = \{(u, v) \in E \mid u \text{ is a smaller neighbor of } v\}$.
   For $v = s$, define $Foot(s) = \{(s, t)\}$, $Hand(s) = \{(s, u) \in E \mid u \neq t\}$. (Note, the two definitions for the source s are special.)
2. *For face $v^* \neq t^*$ of G^*, define:*
   $cover(v^*) = \{e^* \mid e \in Hand(v)\}$;
   $sheet(v^*) = \{e^* \mid e \in Foot(v)\}$.
   Note that, for any face $v^* \neq t^*$ of G^* (including s^*), its boundary consists of two directed paths, one is $cover(v^*)$, the other is $sheet(v^*)$.

3. For vertex $v \neq t$ of G, define $score_{\mathcal{O}}(v) = \min\{|Hand(v)|,\ |Foot(v)|\}$, and $Score_{\mathcal{O}}(G) = \sum_{v \neq t} score_{\mathcal{O}}(v)$.

For example, in Fig. 2, $Hand(v_3) = \{(v_3, v_4),\ (v_3, v_5)\}$ and $Foot(v_3) = \{(s, v_3),\ (v_2, v_3)\}$. For v_3^* in G^*, $cover(v_3^*) = \{(f_1, f_3),\ (f_3, f_4)\}$ and $sheet(v^*) = \{(f_1, f_2),\ (f_2, f_4)\}$.

Now we can give the following technical theorem:

Theorem 1 *Let G be a 2-connected plane graph with an st-orientation $\mathcal{O}$. Let $\mathcal{O}^*$ be the st-orientation of G^* derived from $\mathcal{O}$. Then G has a VR with width at most $|E| - Score_{\mathcal{O}}(G)$.*

Proof: We embed G and G^* in the plane, such that the face t^* of G^* as the exterior face of G^*. Let

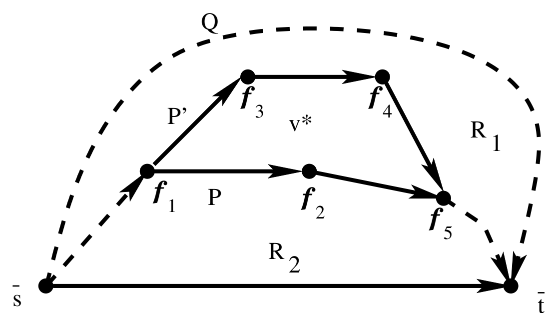

Figure 3: The proof of Theorem 1.

$(\overline{s}, \overline{t})$ be the distinguished exterior edge of $\mathcal{O}^*$, Q be the other path connecting $\overline{s}, \overline{t}$ which is part of the exterior boundary. Then Q and $(\overline{s}, \overline{t})$ enclose a closed region, denoted by R, consisting of all the interior faces of G^* (namely, all the dual faces $v^* \neq t^*$ of $v \neq t$ in G). See Fig. 3 for an illustration.

Let P be the longest directed path in $\mathcal{O}^*$ from $\overline{s}$ to $\overline{t}$. P cut the region R into two subregions: R_1, R_2, where R_1 is enclosed by Q and P, R_2 is enclosed by P and $(\overline{s}, \overline{t})$. We observe that for any face v^* inside R_1 (R_2, resp.), its $cover(v^*)$ ($sheet(v^*)$, resp.) is not on the path P. For example, in Fig. 3, v^* *is in* R_1, $cover(v^*) = \{(f_1, f_3),\ (f_3, f_4),\ (f_4, f_5)\}$ is not on P. Also observe that: each edge in G^* can only be in at most one cover set. Thus: $P \cap (\cup_{\{v^* \ in \ R_1\}} cover(v^*)) = \emptyset$. Similarly, $P \cap (\cup_{\{v^* \ in \ R_2\}} sheet(v^*)) = \emptyset$.

Note that all the interior faces of G^* are either in R_1, or in R_2. So, $\sum_{\{v^* \ in \ R_1\}} |cover(v^*)| + \sum_{\{v^* \ in \ R_2\}} |sheet(v^*)| = \sum_{\{v| \ v^* \ in \ R_1\}} |Hand(v)| + \sum_{\{v| \ v^* \ in \ R_2\}} |Foot(v)| \geq Score_{\mathcal{O}}(G)$. Thus, the length of P is at most $|E| - Score_{\mathcal{O}}(G)$. By Lemma 1, G has a VR with width at most $|E| - Score_{\mathcal{O}}(G)$. □

In [5], Lin et. al. proved that their sophisticated greedy algorithm outputs a VR of G with the width bounded by $|E| - Score_{\mathcal{O}}(G)$. Theorem 1 shows that the simpler VR algorithm in [6, 9] achieves the same width bound.

For a 4-connected plane triangulation G, Kant and He gave an *st*-orientation $\mathcal{O}$ of G such that $Score_{\mathcal{O}}(G) \geq (2n - 5)$ [3]. Thus, using this *st*-orientation, Algorithm 1 outputs a VR of G with width at most $(n - 1)$.

4 Compact Visibility Representation

Thus, in order to shorten the width of a VR of a plane graph G, we need to find an *st*-orientation $\mathcal{O}$ of G such that $Score_{\mathcal{O}}(G)$ is as large as possible.

Without loss of generality, we assume that G is a plane triangulation with $n \geq 4$ vertices in this section. First we need to introduce the concept of Schnyder's realizer [7, 8]:

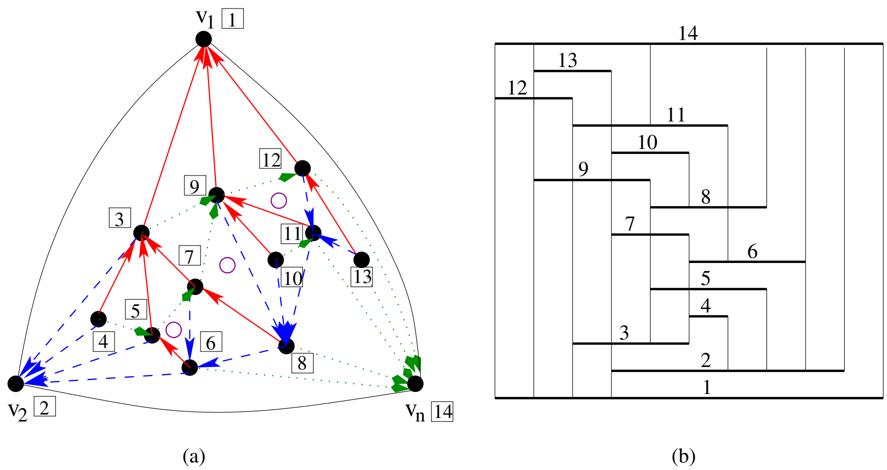

Figure 4: (a) A plane triangulation G and one realizer $\mathcal{R}$ of G; (b) A VR of G.

Definition 2 *Let G be a plane triangulation with three exterior vertices v_1, v_2, v_n in ccw order. A* realizer $\mathcal{R}$ *of G is a partition of the interior edges of G into three sets T_1, T_2, T_n of directed edges such that the following hold:*

- *For each $i \in \{1, 2, n\}$, the interior edges incident to v_i are in T_i and directed toward v_i.*
- *For each interior vertex v of G, the neighbors of v form six blocks $U_1, D_n, U_2, D_1, U_n,$ and D_2 in ccw order around v, where U_j and D_j $(j = 1, 2, n)$ are the parent and the children of v in T_j.*

It was shown in [7, 8] that every plane triangulation G has a realizer $\mathcal{R}$, which can be obtained in linear time. Each T_i $(i \in \{1, 2, n\})$ is a tree rooted at the vertex v_i containing all interior vertices of G. Fig. 4 (a) shows a realizer of a plane triangulation G. Three trees T_1, T_2, T_n are drawn as solid, dashed, and dotted lines, respectively. (Ignore the small boxes containing integers for now. Their meaning will be explained later.)

The following lemma shows how to obtain st-numberings from a Schnyder's realizer [10].

Lemma 2 *Let G be a plane triangulation and $\mathcal{R} = \{T_1, T_2, T_n\}$ a Schnyder's realizer of G. T_i' be the tree obtained by T_i plus the two exterior edges adjacent to v_i in G. T_i' is rooted at v_i. Then the ccw preordering of the vertices of G with respect to T_i' is an st-numbering of G.*

For example, consider the tree T_1 (rooted at v_1) shown in Fig. 4. The union of T_1 and the two exterior edges (v_2, v_1) and (v_n, v_1) is a tree of G, denote it by T_1'. The ccw preordering of the vertices of G with respect to T_1' are shown in integers inside the small boxes. It is an st-orientation of G by Lemma 2, denoted by $\mathcal{O}_1$. Similarly, we have two other st-orientations $\mathcal{O}_2$, $\mathcal{O}_n$.

Denote the set of interior vertices of G by I. Then for each vertex $v \in I$, $score_{\mathcal{O}_i}(v), i = 1, 2, n$ is always definable. And obviously $Score_{\mathcal{O}_i}(G) = 2 + \sum_{v \in I} score_{\mathcal{O}_i}(v)$. We denote $score_{sum}(v) = \sum_{i=1,2,n} score_{\mathcal{O}_i}(v)$ for each $v \in I$.

Next, we want to find a lower bound of $\sum_{i=1,2,n} Score_{\mathcal{O}_i}(G)$.

Let $inter(v) = \sum_{i=1,2,n} [v \text{ is not a leaf of } T_i]$, where $[c]$ is 1 (0, resp.) if condition c is true (false, resp.). Lin et. al. partitioned the interior vertices of G into three subsets A, B, C as follows [5]:

$A = \{v |\ inter(v) = 0\}$;

$B = \{v |\ inter(v) = 2,\ deg(v) = 5\}$;

$C = \{v \notin B |\ inter(v) \geq 1\}$.

Let ξ_i be the number of internal (namely, non-leaf) vertices in T_i. An interior face f of G is *cyclic* with respect to R if each of its three edges belongs to different trees of R. Denote the number of cyclic interior faces with respect to R by $\Delta(\mathcal{R})$. For example, in Fig. 4 (a), the faces $\{5,7,6\}$, $\{7,9,8\}$, $\{9,12,11\}$ (marked by empty circles) are the cyclic faces in $\mathcal{R}$. So, $\Delta(\mathcal{R}) = 3$.

The following results were proved in [1, 5]:

Lemma 3 *Let G be a plane triangulation with $n \geq 4$ vertices. Let v_1, v_2, v_n be the exterior vertices of G in the ccw order. Let $\mathcal{R} = \{T_1, T_2, T_n\}$ be any realizer of G, where T_i, $i \in \{1, 2, n\}$ is rooted at v_i respectively. Let k be the number of connected components of the graph $G[B]$, which is a subgraph of G induced by B.*

1. $\xi_1 + \xi_2 + \xi_n - \Delta(\mathcal{R}) = n - 1.$
2. $\xi_1 + \xi_2 + \xi_n - 3 = \sum_{v \in I} inter(v) \geq 2|B| + |C|.$
3. $score_{sum}(v) \geq 3 + 2 \cdot inter(v) - [v \in B],\ v \in\ I.$
4. $|B| - k \leq 2\Delta(\mathcal{R}).$

Now we can prove the following theorem:

Theorem 2 *Let G be a plane triangulation with $n \geq 4$ vertices, $R = \{T_1, T_2, T_n\}$ be any Schnyder's realizer of G. Then $\sum_{i=1,2,n} Score_{\mathcal{O}_i}(G) \geq \frac{14n}{3} - 10$.*

Proof: Let $G[B]$ be the subgraph of G induced by B. Suppose that $G[B]$ has k connected components. Let $|B| - k = \delta\Delta(\mathcal{R})$, then we have $0 \leq \delta \leq 2$ by Lemma 3 (4). Let B_t, $t = 1, 2, \cdots, k$ be all the connected components of $G[B]$. Lin et. al. observed that [5]: any two distinct vertices of A are not adjacent in G; and each vertex in A is adjacent to at most one B_t. Thus, we know that the number of the connected components of $G[A \cup B]$ is at least k. Considering $G - (A \cup B)$, each interior face of $G - (A \cup B)$ contains at most one connected component of $G[B]$. We remove edges of $G - (A \cup B)$ until each interior face contains exactly one connected component of $G[B]$, denote this graph by G'. (If $G[B]$ is empty, G' does not have interior face.) Let F_i $(i = 3, 4, \cdots)$ be the set of interior faces of G' with i edges on its boundary. Thus, we have:

$$k = \sum_{i=3}^{\infty} |F_i| \tag{1}$$

For any face in F_i, $i \geq 4$, we can triangulate it into $i - 2$ faces, then by applying Euler's formula to the resulting graph, its number of interior faces is at most $2(|C| + 3) - 5 = 2|C| + 1$. (3 comes from the exterior vertices of G.) Thus:

$$\sum_{i=3}^{\infty} (i-2)|F_i| \leq 2(|C| + 3) - 5 = 2|C| + 1.$$

Therefore:

$$\sum_{i=3}^{\infty} \frac{i-2}{2}|F_i| - \frac{1}{2} \leq |C| \tag{2}$$

Using Equation (1) and (2), we have:

$$|C| \geq \frac{k}{2} + \frac{1}{2}|F_4| + |F_5| + \frac{1}{2}|F_6| - \frac{1}{2} \tag{3}$$

Applying Lemma 3 (1) and (2) and above equation, we have:

$$\begin{aligned} n+\Delta(\mathcal{R})-4 \geq 2|B|+|C| &\geq 2k+2\delta\Delta(\mathcal{R})+\frac{k}{2}+\frac{1}{2}|F_4|+|F_5|+\frac{1}{2}|F_6|-\frac{1}{2} \\ &= \frac{5}{2}k+2\delta\Delta(\mathcal{R})-\frac{1}{2}+\frac{1}{2}|F_4|+|F_5|+\frac{1}{2}|F_6|. \end{aligned}$$

Thus, we have:

$$\frac{5}{2}k \leq n+\Delta(\mathcal{R})-2\delta\Delta(\mathcal{R})-\frac{7}{2}-\frac{1}{2}|F_4|-|F_5|-\frac{1}{2}|F_6| \tag{4}$$

Because any vertex in B has degree 5 in G, and an interior face of G' in F_3 contains at least 1 vertex from B in G, so it contains at least 3 vertices from $A \cup B$ in G. Similarly, an interior face of G' in F_4 contains at least 2 vertices from $A \cup B$ in G. An interior face of G' in F_i for $i \geq 5$ contains at least 1 vertices from $A \cup B$ in G. Thus:

$$3|F_3|+2|F_4|+\sum_{i=5}^{\infty}|F_i| \leq |A|+|B|.$$

Add this to Equation (2), we have:

$$\frac{7}{2}|F_3|+3|F_4|+\frac{5}{2}|F_5|+3|F_6|+\sum_{i=7}^{\infty}\frac{i}{2}|F_i| \leq |A|+|B|+|C|+\frac{1}{2}=n-\frac{5}{2}.$$

Combining it with Equation (1), we have:

$$\begin{aligned} \frac{7}{2}k &\leq (\frac{7}{2}|F_3|+3|F_4|+\frac{5}{2}|F_5|+3|F_6|+\sum_{i=7}^{\infty}\frac{i}{2}|F_i|)+(\frac{1}{2}|F_4|+|F_5|+\frac{1}{2}|F_6|) \\ &\leq n-\frac{5}{2}+\frac{1}{2}|F_4|+|F_5|+\frac{1}{2}|F_6| \end{aligned} \tag{5}$$

Add Equation (4) and (5), and divide both sides by 6. We have:

$$k \leq \frac{n}{3}+\frac{1}{6}\Delta(\mathcal{R})-\frac{1}{3}\delta\Delta(\mathcal{R})-1 \tag{6}$$

Applying Lemma 3 (2), (3) and above equation, also note that $0 \leq \delta \leq 2$, we have:

$$\begin{aligned} \sum_{i=1,2,n} Score_{\mathcal{O}_i}(G) &= 6+\sum_{v\in I} score_{sum}(v) \\ &\geq 6+\sum_{v\in I}\{3+2\cdot inter(v)\}-|B| \\ &= 6+3(n-3)+2(n+\Delta(\mathcal{R})-4)-|B| \\ &= 5n+2\Delta(\mathcal{R})-11-|B| \\ &= 5n+2\Delta(\mathcal{R})-11-(|B|-k)-k \\ &\geq 5n+2\Delta(\mathcal{R})-11-\delta\Delta(\mathcal{R})-\frac{1}{3}n-\frac{1}{6}\Delta(\mathcal{R})+\frac{1}{3}\delta\Delta(\mathcal{R})+1 \\ &= \frac{14n}{3}+(\frac{11}{6}-\frac{2}{3}\delta)\Delta(\mathcal{R})-10 \geq \frac{14n}{3}-10 \end{aligned} \tag{7}$$

□

Theorem 3 *Let G be a plane triangulation with $n \geq 4$ vertices, then G has a VR with width at most $\lfloor \frac{13n-24}{9} \rfloor$.*

Proof: Applying Theorem 2, we have $\sum_{i=1,2,n} Score_{O_i}(G) \geq \frac{14n}{3} - 10$. Thus, one of $Score_{O_i}(G) \geq \lceil \frac{14n-30}{9} \rceil$. Applying Theorem 1 and Algorithm 1, the width of the VR is at most $3n - 6 - \lceil \frac{14n-30}{9} \rceil = \lfloor \frac{13n-24}{9} \rfloor$. □

For example, Fig. 4 (b) gives a VR of G, using the st-numbering in Fig. 4 (a).

Whether the bound $\frac{13n}{9} - O(1)$ is worst case optimal remains open.

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